A characterization of open modalities in homotopy type theory

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1 Introduction

In this note, we prove the following theorem in homotopy type theory [4].

Theorem (Theorem 3.1)**.** *A lex, accessible modality is an open modality if and only if its reflector preserves dependent function types.*

This is analogous to the classical characterization of open subtoposes in topos theory [1, A4.5.1]: a subtopos of a topos is an open subtopos if and only if its reflector preserves exponentials.

Our proof is, though, not identical to the topos-theoretic one for two reasons. First, there is no universe of all propositions in homotopy type theory unless propositional resizing [4, Section 3.5] is assumed, so we cannot mimic topos-theoretic arguments using the subobject classifier. Second, even when propositional resizing is assumed, not all modalities are induced by local operators. A typical example is the hypercompletion modality [3, Theorem 3.21] valid in some ∞-toposes of sheaves [2, Section 6.5.4].

Our proof idea comes from the *adjoint functor theorem* for presentable (∞-)categories. The reflector of a lex, accessible modality is considered as a lex, accessible endofunctor on the ∞-category of types. Our main theorem essentially asserts that such a reflector is representable if and only if it preserves products. This should follow from some form of adjoint functor theorem in homotopy type theory. At the moment of writing, no adjoint functor theorem is available in homotopy type theory because even ∞-category theory in homotopy type theory has not yet been developed. However, the proof of the classical adjoint functor theorem tells us how to construct the representing object, and this is enough for our special case.

2 Modalities in homotopy type theory

We review the theory of modalities in homotopy type theory [3].

Definition 2.1. A *modality* $\mathtt{m}$ (on a universe $\mathcal{U}$) is a family of propositions $\mathsf{In}_{\mathtt{m}} : \mathcal{U} \to \mathsf{Prop}_{\mathcal{U}}$ satisfying the following axioms:

1. $\mathsf{In}_\mathfrak{m}$ is *reflective*, that is, there exist a function $\Box_\mathfrak{m} : \mathcal{U} \to \mathcal{U}$ and a family of functions $\eta_\mathfrak{m} : \prod_{A:\mathcal{U}} A \to \Box_\mathfrak{m}(A)$ satisfying that $\mathsf{In}_\mathfrak{m}(\Box_\mathfrak{m}(A))$ for all $A : \mathcal{U}$ and that, for any types $A, B : \mathcal{U}$ such that $\mathsf{In}_\mathfrak{m}(B)$, the function $\lambda f.f \circ \eta_\mathfrak{m}(A) : (\Box_\mathfrak{m}(A) \to B) \to (A \to B)$ is an equivalence;

2. $\mathsf{In}_\mathfrak{m}$ is closed under dependent pair types, that is, for any $A : \mathcal{U}$ and $B : A \to \mathcal{U}$, if $\mathsf{In}_\mathfrak{m}(A)$ and $\prod_{a:A} \mathsf{In}_\mathfrak{m}(B(a))$, then $\mathsf{In}_\mathfrak{m}(\sum_{a:A} B(a))$.

We call $\Box_\mathfrak{m}$ the *reflector* of $\mathfrak{m}$.

Remark 2.2. $\Box_\mathfrak{m}$ and $\eta_\mathfrak{m}$ are uniquely determined by $\mathsf{In}_\mathfrak{m}$.

Remark 2.3. The reflector gives the following $\mathfrak{m}$-*recursion principle*: for any types $A, B : \mathcal{U}$ such that $\mathsf{In}_\mathfrak{m}(B)$ and a function $f : A \to B$, there exists a unique function $\mathsf{rec}_\mathfrak{m}(f) : \Box_\mathfrak{m}(A) \to B$ equipped with an identification $\mathsf{rec}_\mathfrak{m}(f) \circ \eta_\mathfrak{m}(A) = f$.

Definition 2.4. Let $\mathfrak{m}$ be a modality. A type $A : \mathcal{U}$ is $\mathfrak{m}$-*connected* if $\Box_\mathfrak{m}(A)$ is contractible.

Proposition 2.5 ([3, Corollary 1.37])**.** *Let $\mathfrak{m}$ be a modality. A type $A : \mathcal{U}$ is $\mathfrak{m}$-connected if and only if, for any $B : \mathcal{U}$ such that $\mathsf{In}_\mathfrak{m}(B)$, the function $\lambda b.\lambda\_.b : B \to (A \to B)$ is an equivalence.*

Definition 2.6. A modality is *left exact* or *lex* for short if, for any $A : \mathcal{U}$, if A is $\mathfrak{m}$-connected, then so is $a = b$ for any $a, b : A$.

Definition 2.7. By a *family* we mean a pair (I_α, Z_α) consisting of $I_\alpha : \mathcal{U}$ and $Z_\alpha : I_\alpha \to \mathcal{U}$.

Definition 2.8. Let $\mathfrak{m}$ be a modality. A *presentation of* $\mathfrak{m}$ is a family α such that a type $A : \mathcal{U}$ satisfies $\mathsf{In}_\mathfrak{m}(A)$ if and only if the function $\lambda a.\lambda\_.a : A \to (Z_\alpha(i) \to A)$ is an equivalence for all $i : I_\alpha$.

Remark 2.9. If α is a presentation of $\mathfrak{m}$, then $Z_\alpha(i)$ is $\mathfrak{m}$-connected for any $i : I_\alpha$ by Proposition 2.5.

Definition 2.10. A modality is *accessible* if it admits a presentation.

Example 2.11. Let $P : \mathsf{Prop}_\mathcal{U}$. We define the *open modality* $\mathfrak{o}_P$ by

$$\mathsf{In}_{\mathfrak{o}_P}(A) \equiv \mathsf{IsEquiv}(\lambda(a : A).\lambda(\_ : P).a).$$

The reflector is defined by $\Box_{\mathfrak{o}_P}(A) \equiv (P \to A)$ with unit $\eta_{\mathfrak{o}_P}(A, a) \equiv \lambda\_.a$. The modality $\mathfrak{o}_P$ is accessible since $I_\alpha \equiv \mathsf{Unit}$ and $Z_\alpha(\_) \equiv P$ define a presentation of $\mathfrak{o}_P$. One can see that a type $A : \mathcal{U}$ is $\mathfrak{o}_P$-connected if and only if $P \to \mathsf{IsContr}(A)$, from which it follows that $\mathfrak{o}_P$ is lex.

In Proposition 2.14 below, we give an explicit account of part of [3, Remark 3.23].

Definition 2.12. We say a family α is *closed under identity types* if it is equipped with a function

$$\prod_{i:I_\alpha} \prod_{x,y:Z_\alpha(i)} \sum_{j:I_\alpha} Z_\alpha(j) \simeq (x = y).$$

Construction 2.13. Let α be a family. We define another family $\delta(\alpha)$ by

$$\begin{aligned} I_{\delta(\alpha)} &\equiv I_\alpha + (\textstyle\sum_{i:I_\alpha} Z_\alpha(i) \times Z_\alpha(i)) \\ Z_{\delta(\alpha)}(\mathsf{inl}(i)) &\equiv Z_\alpha(i) \\ Z_{\delta(\alpha)}(\mathsf{inr}(i,x,y)) &\equiv (x = y). \end{aligned}$$

This gives an endofunction δ on the type of families. We then define a family $\delta^*(\alpha)$ by

$$\begin{aligned} I_{\delta^*(\alpha)} &\equiv \textstyle\sum_{n:\mathsf{Nat}} I_{\delta^n(\alpha)} \\ Z_{\delta^*(\alpha)}(n,i) &\equiv Z_{\delta^n(\alpha)}(i). \end{aligned}$$

By construction, $\delta^*(\alpha)$ is closed under identity types: given $(n,i) : I_{\delta^*(\alpha)}$ and $x, y : Z_{\delta^*(\alpha)}(n,i)$, we have $(n+1, \mathsf{inr}(i,x,y)) : I_{\delta^*(\alpha)}$ and the identity equivalence $Z_{\delta^*(\alpha)}(n+1, \mathsf{inr}(i,x,y)) \simeq (x = y)$.

Proposition 2.14. *Any lex, accessible modality* $\mathfrak{m}$ *admits a presentation closed under identity types.*

Proof. Let α be a presentation of $\mathfrak{m}$. Then $\delta(\alpha)$ is also a presentation of $\mathfrak{m}$. To see this, let $A : \mathcal{U}$ be a type. Since $\delta(\alpha)$ includes α, if the function $A \to (Z_{\delta(\alpha)} \to A)$ is an equivalence for all $i : I_{\delta(\alpha)}$, then $\mathsf{In}_\mathfrak{m}(A)$. Conversely, if $\mathsf{In}_\mathfrak{m}(A)$, then the function $A \to ((x = y) \to A)$ is an equivalence for all $\mathsf{inr}(i,x,y) : I_{\delta(\alpha)}$ by Proposition 2.5 since $x = y$ is $\mathfrak{m}$-connected as $\mathfrak{m}$ is lex. Therefore, $\delta^n(\alpha)$'s are all presentations of $\mathfrak{m}$, and then $\delta^*(\alpha)$ is a presentation of $\mathfrak{m}$ as well and closed under identity types. ☐

3 Main theorem

Theorem 3.1. *A lex, accessible modality is an open modality if and only if its reflector preserves dependent function types.*

We clarify what preservation of dependent function types means.

Proposition 3.2 ([3, Lemma 1.26])**.** *Let* $\mathfrak{m}$ *be a modality. For* $A : \mathcal{U}$ *and* $B : A \to \mathcal{U}$*, if* $\prod_{a:A} \mathsf{In}_\mathfrak{m}(B(a))$*, then* $\mathsf{In}_\mathfrak{m}(\prod_{a:A} B(a))$.

Definition 3.3. Let $\mathfrak{m}$ be a modality. For $A : \mathcal{U}$ and $B : A \to \mathcal{U}$, we have a canonical function

$$\mathsf{c}^\Pi_\mathfrak{m} : \Box_\mathfrak{m}(\textstyle\prod_{a:A} B(a)) \to \prod_{a:A} \Box_\mathfrak{m}(B(a))$$

with an identification $\mathsf{c}^\Pi_\mathfrak{m}(\eta_\mathfrak{m}(\_, f)) = \lambda a.\eta_\mathfrak{m}(\_, f(a))$ for $f : \prod_{a:A} B(a)$. This is defined by $\mathfrak{m}$-recursion since $\mathsf{In}_\mathfrak{m}(\prod_{a:A} \Box_\mathfrak{m}(B(a)))$ by Proposition 3.2. We say $\Box_\mathfrak{m}$ *preserves dependent function types* if the function $\mathsf{c}^\Pi_\mathfrak{m}$ is an equivalence for all A and B.

The following is immediate from the definition.

Lemma 3.4. *Let $\mathfrak{m}$ be a modality whose reflector preserves dependent function types. For $A : \mathcal{U}$ and $B : A \to \mathcal{U}$, if $B(a)$ is $\mathfrak{m}$-connected for all $a : A$, then so is $\prod_{a:A} B(a)$.* □

Our main theorem will follow from the following technical lemma.

Lemma 3.5. *Let α be a family closed under identity types. Then*

$$\left(\prod_{i:I_\alpha} Z_\alpha(i)\right) \to \prod_{i:I_\alpha} \mathsf{IsContr}(Z_\alpha(i)).$$

Proof. Suppose $f : \prod_{i:I_\alpha} Z_\alpha(i)$ and let $i : I_\alpha$. Then $Z_\alpha(i)$ is inhabited by $f(i)$. For any $x : Z_\alpha(i)$, we have some $j : I_\alpha$ and $e : Z_\alpha(j) \simeq (f(i) = x)$ since α is closed under identity types. Then $e(f(j)) : f(i) = x$. □

Corollary 3.6. *Let α be a family closed under identity types. Then the type*

$$\prod_{i:I_\alpha} Z_\alpha(i)$$

is a proposition.

Proof. Lemma 3.5 implies $\left(\prod_{i:I_\alpha} Z_\alpha(i)\right) \to \mathsf{IsContr}\left(\prod_{i:I_\alpha} Z_\alpha(i)\right)$. □

Construction 3.7. Let $\mathfrak{m}$ be a lex, accessible modality. Take a presentation α of $\mathfrak{m}$ closed under identity types by Proposition 2.14. We define

$$\Phi_\mathfrak{m} \equiv \prod_{i:I_\alpha} Z_\alpha(i)$$

which is a proposition by Corollary 3.6.

$\mathfrak{o}_{\Phi_\mathfrak{m}}$ is the best approximation of $\mathfrak{m}$ by an open modality:

Proposition 3.8. *Let $\mathfrak{m}$ be a lex, accessible modality. The proposition $\Phi_\mathfrak{m}$ in Construction 3.7 is the largest proposition in $\mathcal{U}$ such that $\mathsf{In}_{\mathfrak{o}_{\Phi_\mathfrak{m}}}(A) \to \mathsf{In}_\mathfrak{m}(A)$ for all $A : \mathcal{U}$. Consequently, $\Phi_\mathfrak{m}$ does not depend on the choice of presentation of $\mathfrak{m}$.*

Proof. Consider the following commutative diagram for $i : I_\alpha$ and $A : \mathcal{U}$.

$$\begin{array}{ccc} A & \longrightarrow & (Z_\alpha(i) \to A) \\ \downarrow & & \downarrow \\ (\Phi_\mathfrak{m} \to A) & \xrightarrow[\simeq]{} & (Z_\alpha(i) \to \Phi_\mathfrak{m} \to A) \end{array}$$

The bottom function is an equivalence by Lemma 3.5. The vertical functions are equivalences when $\mathsf{In}_{\mathfrak{o}_{\Phi_\mathfrak{m}}}(A)$. Then, by 2-out-of-3, $\mathsf{In}_{\mathfrak{o}_{\Phi_\mathfrak{m}}}(A)$ implies $\mathsf{In}_\mathfrak{m}(A)$.

Let $P : \mathsf{Prop}_\mathcal{U}$ be a proposition such that $\mathsf{In}_{\mathfrak{o}_P}(A) \to \mathsf{In}_\mathfrak{m}(A)$ for all $A : \mathcal{U}$. By Proposition 2.5, any $\mathfrak{m}$-connected type is $\mathfrak{o}_P$-connected. In particular, $Z_\alpha(i)$ is $\mathfrak{o}_P$-connected for any i, that is $P \to \mathsf{IsContr}(Z_\alpha(i))$, from which we have $P \to \prod_{i:I_\alpha} Z_\alpha(i)$. □

Proof of Theorem 3.1. The "only if" part can directly be verified. For the "if" part, let $\mathfrak{m}$ be a lex, accessible modality whose reflector preserves dependent function types. We show that $\mathsf{In}_{\mathfrak{m}}(A)$ if and only if $\mathsf{In}_{\bigcirc_{\Phi_{\mathfrak{m}}}}(A)$, for any $A : \mathcal{U}$. We have seen in Proposition 3.8 that $\mathsf{In}_{\bigcirc_{\Phi_{\mathfrak{m}}}}(A)$ implies $\mathsf{In}_{\mathfrak{m}}(A)$. By Lemma 3.4, $\Phi_{\mathfrak{m}}$ is $\mathfrak{m}$-connected, and then $\mathsf{In}_{\mathfrak{m}}(A)$ implies $\mathsf{In}_{\bigcirc_{\Phi_{\mathfrak{m}}}}(A)$ by Proposition 2.5. $\square$

Remark 3.9. What is happening in the proof of Theorem 3.1 is a special case of Freyd's *adjoint functor theorem*. The reflector $\square_{\mathfrak{m}} : \mathcal{U} \to \mathcal{U}$ is thought of as a lex, accessible functor, and Theorem 3.1 says that it is representable by some type $\Phi_{\mathfrak{m}} : \mathcal{U}$ (which is automatically a proposition since $\square_{\mathfrak{m}}$ is an idempotent monad) if and only if it preserves products. The type family $Z_\alpha : I_\alpha \to \mathcal{U}$ is thought of as a "solution set", and $\Phi_{\mathfrak{m}}$ is constructed as the "limit" of Z_α. Note that the product $\Phi_{\mathfrak{m}} \equiv \prod_{i:I_\alpha} Z_\alpha(i)$ is the limit of any considerable extension of Z_α to a functor, because α is closed under identity types. Indeed, for any $i, j : I_\alpha$ and any function $f : Z_\alpha(i) \to Z_\alpha(j)$, the diagram

$$\begin{array}{ccc} & \Phi_{\mathfrak{m}} & \\ \swarrow & & \searrow \\ Z_\alpha(i) & \xrightarrow[f]{} & Z_\alpha(j) \end{array}$$

commutes by Lemma 3.5.

Remark 3.10. Assume propositional resizing so we have the universe Prop of all propositions. Suppose that $\mathfrak{m}$ is a topological modality. By [3, Theorem 3.37], $\mathfrak{m}$ is induced by a local operator j, and we can choose a canonical persentation α of $\mathfrak{m}$ defined by $I_\alpha \equiv \{P : \mathsf{Prop} \mid P \text{ is } j\text{-dense}\}$ and $Z_\alpha(P) \equiv P$. Then the construction of the proposition $\Phi_{\mathfrak{m}}$ coincides with the construction of the subterminal object in the proof of [1, A4.5.1].

References

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