Homotopy type theory as internal languages of diagrams of ∞-logoses

Taichi Uemura

July 4, 2023
FSCD 2023

Homotopy type theory (HoTT)

Syntax	Semantics
HoTT	an ∞-logos (a.k.a. ∞-topos)
?	a diagram of ∞-logoses

Contributions I

- Introduce a notion of a *mode sketch*.
- A diagram of ∞-logoses indexed over a mode sketch is reconstructed via *modalities* in its *oplax limit*.

Syntax	Semantics
HoTT	an ∞-logos
HoTT+modalities	a diagram of ∞-logoses indexed over a *mode sketch*

Remarks on modalities

- By modalities we mean modalities *internal* to HoTT (i.e. the type of modalities is defined in HoTT).
- We postulate some modalities as *constants* rather than rules.
- Can be formalized in any HoTT library.
- Easy to use also informally.

Contributions II

HoTT+modalities is also a language for *higher dimensional logical relations* analogous to *synthetic Tait computability (STC)* (Sterling and Harper 2021).

Motivation: ∞-type theories

Normalization for ∞-type theories (Uemura 2022) is an approach to general coherence theorems in the ∞-categorical semantics of type theories.

- Use the STC-viewpoint to construct a normalization model to get a normalization map.
- Use the internal-diagram-viewpoint to analyze the normalization map.

Modalities in HoTT

Studied by Rijke, Shulman, and Spitters (2020).

Definition

A *lex, accessible modality (LAM)* (on $\mathcal{U}$) $\mathfrak{m}$ consists of a $\mathsf{In}_{\mathfrak{m}} : \mathcal{U} \to \mathsf{Prop}$ such that:

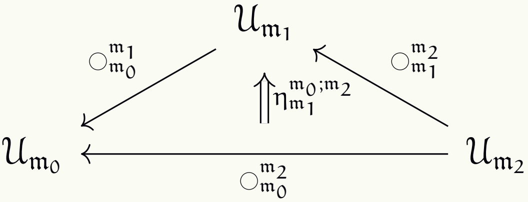
- $\mathcal{U}_{\mathfrak{m}} \equiv \{A : \mathcal{U} \mid \mathsf{In}_{\mathfrak{m}}(A)\} \hookrightarrow \mathcal{U}$ has a left adjoint $\bigcirc_{\mathfrak{m}} : \mathcal{U} \to \mathcal{U}_{\mathfrak{m}}$ with unit $\eta_{\mathfrak{m}} : \prod_{A:\mathcal{U}} A \to \bigcirc_{\mathfrak{m}} A$;
- some other axioms.

Diagrams induced by LAMs

Given two LAMs $\mathfrak{m}_0$ and $\mathfrak{m}_1$, we have a functor

$$\bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_1} : \ \mathcal{U}_{\mathfrak{m}_1} \hookrightarrow \mathcal{U} \xrightarrow{\bigcirc_{\mathfrak{m}_0}} \mathcal{U}_{\mathfrak{m}_0}.$$

Given three LAMs $\mathfrak{m}_0$, $\mathfrak{m}_1$, and $\mathfrak{m}_2$, we have a natural transformation $\eta_{\mathfrak{m}_1}^{\mathfrak{m}_0;\mathfrak{m}_2} \equiv \bigcirc_{\mathfrak{m}_2}^{\mathfrak{m}_1} \eta_{\mathfrak{m}_1}|_{\mathcal{U}_{\mathfrak{m}_2}}$.



Diagrams induced by LAMs

Idea

Postulate some LAMs $\mathfrak{m}(i)$ and use $\bigcirc_{\mathfrak{m}(i)}^{\mathfrak{m}(j)}$'s and $\eta_{\mathfrak{m}(i)}^{\mathfrak{m}(i);\mathfrak{m}(k)}$'s to draw internal diagrams of ∞-logoses.

Trimming

Definition

We write $\mathfrak{m}_0 \leq {}^{\perp}\mathfrak{m}_1$ if $\bigcirc_{\mathfrak{m}_1} A \simeq 1$ whenever $\mathsf{In}_{\mathfrak{m}_0}(A)$.

When $\mathfrak{m}_0 \leq {}^{\perp}\mathfrak{m}_1$, the functor $\bigcirc^{\mathfrak{m}_0}_{\mathfrak{m}_1} : \mathcal{U}_{\mathfrak{m}_0} \to \mathcal{U}_{\mathfrak{m}_1}$ is constant at 1, so *only one direction* $\bigcirc^{\mathfrak{m}_1}_{\mathfrak{m}_0} : \mathcal{U}_{\mathfrak{m}_1} \to \mathcal{U}_{\mathfrak{m}_0}$ *is meaningful*.

Idea

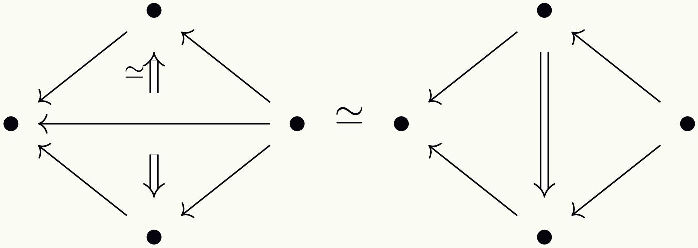
Postulate some $\mathfrak{m}(\mathfrak{i}) \leq {}^{\perp}\mathfrak{m}(\mathfrak{j})$ to cut edges off the diagram.

Commutativity

Idea

Postulate that some $\eta_{\mathfrak{m}(j)}^{\mathfrak{m}(i);\mathfrak{m}(k)}$'s are invertible to make commutative triangles.

Shapes other than triangle, e.g.



Mode sketch

Definition

A *mode sketch* $\mathfrak{M}$ consist of:

- a decidable finite poset $\mathrm{I}_{\mathfrak{M}}$;
- a subset $\mathrm{T}_{\mathfrak{M}}$ of triangles in $\mathrm{I}_{\mathfrak{M}}$ called *thin* triangles.

Mode sketch axioms

Let $\mathfrak{M}$ be a mode sketch and let $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$.

Axiom A $\mathfrak{m}(i) \leq {}^{\perp}\mathfrak{m}(j)$ for any $j \not\leq i$ in $\mathfrak{M}$.

Axiom B $\eta_{\mathfrak{m}(j)}^{\mathfrak{m}(i);\mathfrak{m}(k)}$ is invertible for any thin triangle $(i < j < k)$.

Sometimes we also consider:

Axiom C The join of all $\mathfrak{m}(i)$'s is the top modality.

Models of a mode sketch

Definition

A *model of* $\mathfrak{M}$ is an ∞-logos $\mathcal{L}$ equipped with an interpretation of a constant $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ satisfying Axioms A–C.

∞-logoses indexed over a mode sketch

Definition

A $\mathfrak{M}$*-indexed* ∞*-logos* $\mathcal{K}$ consists of:

- an ∞-logos $\mathcal{K}(\mathrm{i})$ for eveyr $\mathrm{i} \in \mathfrak{M}$;
- a lex, accessible functor $\mathcal{K}(\mathrm{i} < \mathrm{j}) : \mathcal{K}(\mathrm{j}) \to \mathcal{K}(\mathrm{i})$ for every $(\mathrm{i} < \mathrm{j})$ in $\mathfrak{M}$;
- a natural transformation $\mathcal{K}(\mathrm{i} < \mathrm{k}) \Rightarrow \mathcal{K}(\mathrm{i} < \mathrm{j}) \circ \mathcal{K}(\mathrm{j} < \mathrm{k})$ for every triangle $(\mathrm{i} < \mathrm{j} < \mathrm{k})$ in $\mathfrak{M}$ that is invertible when $(\mathrm{i} < \mathrm{j} < \mathrm{k})$ is thin;
- coherence.

Semantics of mode sketch

Theorem

For any mode sketch $\mathfrak{M}$, we have an equivalence between:

1. *the space of models of $\mathfrak{M}$;*
2. *the space of $\mathfrak{M}$-indexed ∞-logoses.*

For $1 \to 2$, the diagram $\{\mathcal{U}_{\mathfrak{m}(\mathfrak{i})}\}_{\mathfrak{i}:\mathfrak{M}}$ is interpreted as an $\mathfrak{M}$-indexed ∞-logos. $2 \to 1$ is constructed by the *oplax limit* of a $\mathfrak{M}$-indexed ∞-logoses.

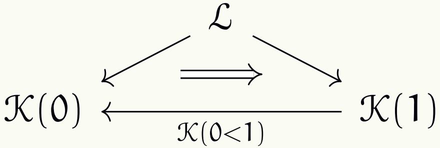
Oplax limit

Example (Oplax limit of an arrow)

Let $\mathfrak{M} \equiv \{0 < 1\}$ and let $\mathcal{K}$ be a $\mathfrak{M}$-indexed ∞-logos. An object in the oplax limit $\mathcal{L} \equiv \operatorname{opLaxLim}_{i \in \mathfrak{M}} \mathcal{K}(i)$ consist of:

- objects $A_0 \in \mathcal{K}(0)$ and $A_1 \in \mathcal{K}(1)$;
- a morphism $A_{0<1} : A_0 \to \mathcal{K}(0 < 1)(A_1)$.

It is part of the *universal oplax cone*.



Oplax limit

Example (Continued)

- The 0th projection $\mathcal{L} \to \mathcal{K}(0)$ has the ff right adjoint $A_0 \mapsto (A_0, \mathbf{1}, !)$ which defines a LAM $\mathfrak{m}(0)$ in $\mathcal{L}$.
- The 1st projection $\mathcal{L} \to \mathcal{K}(1)$ has the ff right adjoint $A_1 \mapsto (\mathcal{K}(0 < 1)(A_1), A_1, \mathbf{id})$ which defines a LAM $\mathfrak{m}(1)$ in $\mathcal{L}$.
- $\bigcirc_{\mathfrak{m}(1)} A_0 \simeq \mathbf{1}$ for $A_0 \in \mathcal{K}(0)$ by definition.

Thus, $(\mathcal{L}, \mathfrak{m})$ is a model of $\mathfrak{M}$.

Synthetic Tait computability

Continue with $\mathfrak{M} \equiv \{0 < 1\}$ and $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ satisfying Axioms A–C.

Theorem (Rijke, Shulman, and Spitters 2020)

$$\mathcal{U} \simeq \sum_{A_0:\mathcal{U}_{\mathfrak{m}(0)}} \sum_{A_1:\mathcal{U}_{\mathfrak{m}(1)}} A_0 \to \bigcirc_{\mathfrak{m}(0)}^{\mathfrak{m}(1)} A_1$$

Corollary

$$\mathcal{U} \simeq \sum_{A_1:\mathcal{U}_{\mathfrak{m}(1)}} A_1 \to \mathcal{U}_{\mathfrak{m}(0)}$$

Types are relations.

Synthetic Tait computability

Let $A : \mathcal{U}$ correspond to $(A_1 : \mathcal{U}_{\mathfrak{m}(1)}, A_0 : A_1 \to \mathcal{U}_{\mathfrak{m}(0)})$.

Proposition

For $A, B : \mathcal{U}$,

$$(A \to B) \simeq \sum_{f:A_1 \to B_1} \prod_{x:A_1} A_0(x) \to B_0(f(x)).$$

This is the formula for $\to$ in the *logical relation translation*/*parametricity translation*.

Types are logical relations.

Conclusion

Introduced the notion of a mode sketch and the mode sketch axioms in HoTT.

- Internal language of diagrams of ∞-logoses.
- Language for logical relations.

References I

E. Rijke, M. Shulman, and B. Spitters (2020). "Modalities in homotopy type theory". In: *Log. Methods Comput. Sci.* 16.1, Paper No. 2, 79. DOI: 10.23638/LMCS-16(1:2)2020.

J. Sterling and R. Harper (2021). "Logical Relations as Types: Proof-Relevant Parametricity for Program Modules". In: *J. ACM* 68.6, 41:1–41:47. DOI: 10.1145/3474834.

T. Uemura (2022). *Normalization and coherence for* ∞*-type theories*. arXiv: 2212.11764v1.

Limitation

Question

Can we add an axiom asserting, say, $\mathcal{K}(0 < 1)$ is ff?

Answer

No, we can't do it naively.

"$\prod_{A,B:\mathcal{U}_{\mathfrak{m}(1)}}(A \to B) \xrightarrow{\simeq} (\bigcirc_{\mathfrak{m}(0)} A \to \bigcirc_{\mathfrak{m}(0)} B)$" asserts that $\mathcal{K}(0 < 1)$ is ff *as a fibred functor over* $\mathcal{L}$ which is stronger than just ff.

Possible solution

Add another LAM expressing "global mode".

Synthetic Tait computability

In Sterling and Harper's synthetic Tait computability, one postulates in ETT a proposition which induces the open and closed modalities.

- In ETT, working with a proposition seems necessary.
- In HoTT, postulating a proposition is in fact equivalent to postulating the mode sketch axioms for $\{0 < 1\}$.