Functor Categories of a Locally Cartesian Closed Category

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Abstract

We show that the functor category from a finite category to a locally cartesian closed category is locally cartesian closed.

1 Locally Cartesian Closed Categories

In this section we review basic theory of locally cartesian closed categories. See also [See84, Section 2] and [Joh02, Section A1.5].

Definition 1. A *locally cartesian closed category* is a finitely complete category $\mathcal{C}$ such that for every object $A \in \mathcal{C}$, the slice category $\mathcal{C}/A$ is cartesian closed.

Definition 2. Let $\mathcal{C}$ be a category. A $\mathcal{C}$*-indexed category* is a pseudo functor [Lac10, Section 3.2] $\mathcal{P} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$. It consists of:

- for each object $A \in \mathcal{C}$, a category $\mathcal{P}(A)$;
- for each morphism $f : A \to B$ in $\mathcal{C}$, a functor $\mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A)$ written simply f^*;
- for each object $A \in \mathcal{C}$, a natural isomorphism $\eta_A : 1_{\mathcal{P}(A)} \Rightarrow 1_A^*$;
- for each morphisms $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$, a natural isomorphism $\mu_{f,g} : f^* g^* \Rightarrow (gf)^*$

such that for any morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$ in $\mathcal{C}$,

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- $\mu_{1_A,f} \circ \eta_A f^* = 1_{f^*}$;
- $\mu_{f,1_B} \circ f^* \eta_B = 1_{f^*}$;
- $\mu_{f,hg} \circ f^* \mu_{g,h} = \mu_{gf,h} \circ \mu_{f,g} h^*$.

Definition 3. Let $\mathcal{C}$ be a finitely complete category. A $\mathcal{C}$-*hyperdoctrine* is a $\mathcal{C}$-indexed category $\mathcal{P}$ such that

1. for each object $A \in \mathcal{C}$, $\mathcal{P}(A)$ is cartesian closed;
2. for each morphism $f : A \to B$ in $\mathcal{C}$, f^* has adjoints $\Sigma_f \dashv f^* \dashv \Pi_f$;
3. for each morphism $f : A \to B$ in $\mathcal{C}$, f^* preserves exponents;
4. $\mathcal{P}$ satisfies the Beck-Chevalley condition: if

$$\begin{array}{ccc} D & \xrightarrow{h} & C \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$

   is a pullback in $\mathcal{C}$, then the canonical natural transformation $\Sigma_k h^* \Rightarrow f^* \Sigma_g$ is an isomorphism.

From the bijections $\mathcal{P}(A)(Z, f^*(X \Rightarrow Y)) \cong \mathcal{P}(B)(\Sigma_f Z \times X, Y)$ and $\mathcal{P}(A)(Z \times f^* X, f^* Y) \cong \mathcal{P}(B)(\Sigma_f (Z \times f^* X), Y)$, the condition 3 is equivalent to

3'. $\mathcal{P}$ satisfies the Frobenius condition: for each morphism $f : A \to B$ in $\mathcal{C}$ and objects $X \in \mathcal{P}(A)$ and $Y \in \mathcal{P}(B)$, the canonical morphism $\Sigma_f (X \times f^* Y) \to \Sigma_f X \times Y$ is an isomorphism.

Example 4. A finitely complete category $\mathcal{C}$ induces a $\mathcal{C}$-indexed category, which we shall denote $\mathcal{C}$ also, given by $\mathcal{C}(A) = \mathcal{C}/A$ where $f^* : \mathcal{C}/B \to \mathcal{C}/A$ is defined by pullback.

The following is a fundamental result on locally cartesian closed categories [See84].

Theorem 5. *Let $\mathcal{C}$ be a finitely complete category. Then $\mathcal{C}$ is locally cartesian closed if and only if the indexed category $\mathcal{C}$ is a hyperdoctrine.*

Proof. The Frobenius condition and the Beck-Chevalley condition are straightforward. For adjoints of a pullback functor, see [Fre72, Section 1.3] or [MLM92, Section I.9]. □

2 Oplax Limits of Indexed Cartesian Closed Categories

The goal of this section is to give a sufficient condition for the oplax limit of an indexed category to be cartesian closed (Proposition 10).

Definition 6. Let $\mathcal{C} : I^{\mathrm{op}} \to \mathbf{Cat}$ be an I-indexed category. The *oplax limit of* $\mathcal{C}$, which we will refer to as $[I, \mathcal{C}]$ in this paper, is the following category.

- The objects consists of:
  - for each object $i \in I$, an object $X_i \in \mathcal{C}(i)$;
  - for each morphism $s : i \to j$ in I, a morphism $X_s : X_i \to s^* X_j$

  such that

  - for any object $i \in I$, $X_{1_i} = \eta_i(X_i)$;
  - for any morphisms $s : i \to j$ and $t : j \to k$ in I, $\mu_{s,t}(X_k) \circ s^* X_t \circ X_s = X_{ts}$.
- The morphisms from X to Y consists of, for each object $i \in I$, a morphism $f_i : X_i \to Y_i$ such that, for any morphism $s : i \to j$ in I, $Y_s \circ f_i = s^* f_j \circ X_s$.

Example 7. If $\mathcal{C}$ is a constant indexed category $i \mapsto \mathcal{C}_0$, then the oplax limit $[I, \mathcal{C}]$ is the functor category $\mathcal{C}_0^I$. Thus oplax limits are a generalization of functor categories.

Definition 8. For a regular cardinal κ, let $\mathbf{CCC}_\kappa$ denote the 2-category of κ-complete cartesian closed categories, functors preserving κ-limits and exponents, and natural transformations. For a category I, an *I-indexed κ-complete cartesian closed category* is a pseudo functor $\mathcal{C} : I^{\mathrm{op}} \to \mathbf{CCC}_\kappa$.

Proposition 9. *Let $\mathcal{C} : I^{\mathrm{op}} \to \mathbf{Cat}$ be an I-indexed category and K a category. Suppose that every $\mathcal{C}(i)$ has K-limits and every $s^* : \mathcal{C}(j) \to \mathcal{C}(i)$ preserves K-limits. Then the oplax limit $[I, \mathcal{C}]$ has K-limits and limits are taken by point-wise limits.*

Proof. Let $X : K \to [I, \mathcal{C}]$ be a K-diagram in $[I, \mathcal{C}]$. Define an object $L \in [I, \mathcal{C}]$ as

- for each $i \in I$, $L_i = \lim_{k \in K} X(k)_i$;

- for each $s : i \to j$, L_s be the composition

$$\lim_k X(k)_i \xrightarrow{\lim_k X(k)_s} \lim_k s^* X(k)_j \xrightarrow{\cong} s^*(\lim_k X(k)_j).$$

It is easy to check that L is a limit of X. $\square$

Before studying exponents in an oplax limit, we recall exponents in the category of set-valued functors. Let I be a category and $X, Y \in \mathbf{Set}^I$. The exponent $X \Rightarrow Y \in \mathbf{Set}^I$ is defined as $(X \Rightarrow Y)_i = \mathbf{Set}^I(I(i,-) \times X, Y)$. For an object $i \in I$, define a functor $T_i : (i \backslash I)^{\mathrm{op}} \times (i \backslash I) \to \mathbf{Set}$ as $T_i(s : i \to j, t : i \to k) = X(j) \Rightarrow Y(k)$. Then $(X \Rightarrow Y)_i$ can be written as the *end* $\int_{s:i \to j} T_i(s,s)$ [ML78, Section IX.5]. Exponents in an oplax limit are a generalization of this formula.

Proposition 10. *Let κ be a regular cardinal and $\mathcal{C} : I^{\mathrm{op}} \to \mathbf{CCC}_\kappa$ an I-indexed κ-complete cartesian closed category. If every coslice category $i \backslash I$ is κ-small, then the oplax limit $[I, \mathcal{C}]$ is κ-complete and cartesian closed.*

Proof. The completeness follows from Proposition 9. To show the cartesian closedness, let $X, Y \in [I, \mathcal{C}]$. For an object $i \in I$, define a functor $T_i : (i \backslash I)^{\mathrm{op}} \times (i \backslash I) \to \mathcal{C}(i)$ as $T_i(s : i \to j, t : i \to k) = s^* X_j \Rightarrow t^* Y_k$. Define an object $E \in [I, \mathcal{C}]$ as

- for each $i \in I$, $E_i = \int_{s:i \to j} T_i(s,s)$;
- for each $f : i \to i'$, E_f is the following composition

  $$\int_{s:i \to j} T_i(s,s) \xrightarrow{\tau} \int_{s:i' \to j} f^* T_{i'}(s,s) \xrightarrow{\cong} f^* \int_{s:i' \to j} T_{i'}(s,s)$$

  where τ is the unique morphism such that

  $$\begin{array}{ccc} \int_{s:i \to j} T_i(s,s) & \xrightarrow{\tau} & \int_{s:i' \to j} f^* T_{i'}(s,s) \\ {\scriptstyle \pi_{sf}} \downarrow & & \downarrow {\scriptstyle \pi_s} \\ (sf)^* X_j \Rightarrow (sf)^* Y_j & \xrightarrow[\cong]{} & f^*(s^* X_j \Rightarrow s^* Y_j) \end{array}$$

  commutes for every $s : i' \to j$.

It is easy to check that E is an exponent of Y by X. $\square$

3 Functor Categories of a Locally Cartesian Closed Category

Definition 11. Let $\mathcal{P} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a $\mathcal{C}$-indexed category and I a category. Define a $\mathcal{C}^I$-indexed category $\mathcal{P}^I$ as follows.

- For each $A \in \mathcal{C}^I$, $\mathcal{P}^I(A) = [I, \mathcal{P}(A_{(-)})]$;
- For each $f : A \to B$ in $\mathcal{C}^I$ and $X \in [I, \mathcal{P}(B_{(-)})]$, $f^*X = (i \mapsto f_i^* X_i)$.

Let $\mathcal{C}$ be a finitely complete category and I a category. There are two $\mathcal{C}^I$-indexed categories $\mathcal{C}^I/-$ and $(\mathcal{C}/-)^I$. For $A \in \mathcal{C}^I$, $(\mathcal{C}^I/-)(A) = \mathcal{C}^I/A$ and $(\mathcal{C}/-)^I(A) = [I, \mathcal{C}/A_{(-)}]$. The following proposition is easy.

Proposition 12. *$\mathcal{C}^I/A$ and $[I, \mathcal{C}/A_{(-)}]$ are naturally isomorphic.*

Now we show the main theorem.

Theorem 13. *Let κ be a regular cardinal, $\mathcal{C}$ a κ-complete locally cartesian closed category and I a category. If every coslice category $i\backslash I$ is κ-small, then $\mathcal{C}^I$ is locally cartesian closed.*

Proof. Let $A \in \mathcal{C}^I$. We show that $\mathcal{C}^I/A$ is cartesian closed. By Proposition 12, it is the oplax limit $[I, \mathcal{C}/A_{(-)}]$. By Proposition 10, it is enough to show that $\mathcal{C}/A_{(-)}$ is an I-indexed κ-complete cartesian closed category. For each $i \in I$, $\mathcal{C}/A_i$ is a κ-complete cartesian closed category by assumption. By Theorem 5, for each $s : i \to j$ in I, $A_s^* : \mathcal{C}/A_j \to \mathcal{C}/A_i$ preserves exponents and has a left adjoint. Since a right adjoint preserves all limits, A_s^* preserves κ-limits. Hence $\mathcal{C}/A_{(-)}$ is an I-indexed κ-complete cartesian closed category. □

Corollary 14. *For a locally cartesian closed category $\mathcal{C}$ and a finite category I, $\mathcal{C}^I$ is locally cartesian closed.*

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