# An elementary definition of opetopic sets

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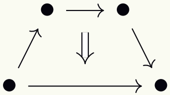
# Opetopes

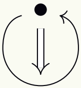
Geometric shapes of many-in-single-out operators in higher dimension. Used for defining weak $\omega$-categories.

0-opetope

1-opetope



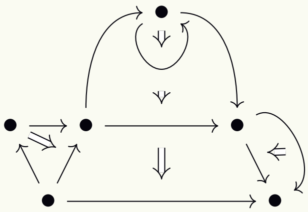
2-opetope with three sources



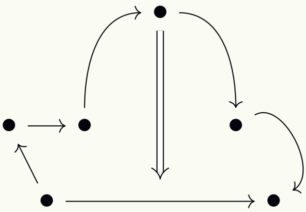
2-opetope with no source

# Opetopes

A $3$-opetope is determined by its pasting diagram of sources.



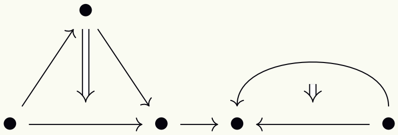
Pasting diagram of sources



Target

# Opetopic sets

The opetopes form a category $\mathbb{O}$. An opetopic set is a set-valued presheaf on $\mathbb{O}$, i.e. a formal colimit of opetopes.



# Formal definitions

- Baez and Dolan [1]
- Leinster [8]
- Kock, Joyal, Batanin, and Mascari [7]
- Hermida, Makkai, and Power [4] (called multitopes and multitopic sets)
- Curien, Ho Thanh, and Mimram [3]

These are not sufficiently accessible: some amount of prerequisites; too long.

# Contribution

I propose elementary definitions of opetopes and opetopic sets.

- Simple structure-axiom style definitions.
- The only prerequisite is basic category theory.
- Less than two pages in A4 size.
- Equivalent to an existing one.

I can explain our definition in full detail in 30 minutes.

# Foundations

We work in Univalent Foundations [9]. Constructively fine: no excluded middle; no choice axiom; no propositional resizing. Non-univalent audience may interpret types as groupoids [6] for this talk.

# Gaunt categories

## Definition

A category is **gaunt** if its type of objects is a set.

In non-univalent foundations, a category is gaunt if the identities are the only isomorphisms in it [2].

# $\omega$-direct categories

## Definition

An $\omega$-**direct category** is a gaunt category $A$ equipped with a conservative functor $\mathbf{deg} : A \to \omega$ called the **degree functor**. A $k$-**step arrow**, written $f : x \to^k y$, is an arrow such that $\mathbf{deg}(x) + k = \mathbf{deg}(y)$. Let $\mathbf{Arr}^k(A)$ denote the set of $k$-step arrows. Let $A \downarrow^k x \subset A \downarrow x$ denote the subset spanned by $k$-step arrows into $x$.

# Preopetopic sets

## Definition

A **preopetopic set** is an $\omega$-direct category $A$ equipped with a subset $\mathcal{S}(A) \subset \mathbf{Arr}^1(A)$ with complement $\mathsf{T}(A)$. A **source arrow**, written $f : x \to^{s} y$, is an arrow in $\mathcal{S}(A)$. A **target arrow**, written $f : x \to^{t} y$, is an arrow in $\mathsf{T}(A)$.

We think of objects in a preopetopic set $A$ as **cells**, and the arrows in $A$ determine the configuration of the cells.

# Opetopic set axioms

An opetopic set is a preopetopic set $A$ satisfying eight axioms.

## Axiom (O1)

$A \downarrow^1 x$ is finite for every $x : A$.

Each cell has finitely many sources and targets.

## Definition

A set $A$ is **finite** if there exist $n : \mathbb{N}$ and $e : \{x : \mathbb{N} \mid x < n\} \simeq A$.

# Opetopic set axioms

## Axiom (O2)

For every object $x : A$ of degree $\geq 1$, there exists a unique target arrow into $x$.

This expresses the single-out nature of opetopes.

## Axiom (O3)

For every object $x : A$ of degree 1, there exists a unique source arrow into $x$.

This expresses that the 1-opetope $(\bullet \to \bullet)$ is single-in.

# Homogeneous/heterogeneous factorizations

## Definition

Let $A$ be a preopetopic set, $f : y \to^1 x$, and $g : z \to^1 y$. We say $(f, g)$ is **homogeneous** if either

- both $f$ and $g$ are source arrows; or
- both $f$ and $g$ are target arrows.

We say $(f, g)$ is **heterogeneous** if either

- $f$ is a source arrow and $g$ is a target arrow; or
- $f$ is a target arrow and $g$ is a source arrow.

By a **homogeneous/heterogeneous factorization** of a 2-step arrow $h$ we mean a factorization $h = f \circ g$ such that $(f, g)$ is homogeneous/heterogeneous.

# Opetopic set axioms

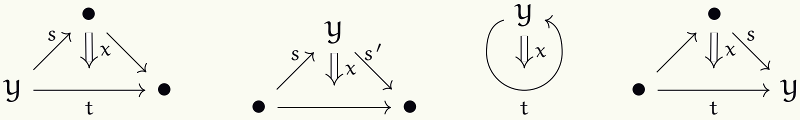
## Axiom (O4)

Every $2$-step arrow in $A$ has a unique homogeneous factorization.

## Axiom (O5)

Every $2$-step arrow in $A$ has a unique heterogeneous factorization.

For example, a $0$-cell $y$ is embedded into a $2$-cell $x$ in exactly two ways, one is homogeneous and the other is heterogeneous.



# Opetopic set axioms

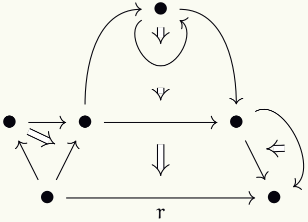
## Axiom (O6)

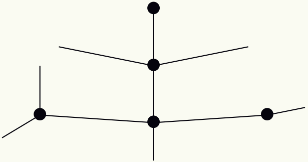
For every object $x : A$ of degree $\geq 2$, there exists a $2$-step arrow $r : A \downarrow^2 x$ such that, for every $2$-step arrow $f : A \downarrow^2 x$, there exists a zigzag

$$f = f_0 \xrightarrow{s_0}{}^{s} g_0 \; {}^{t}\xleftarrow{t_0} f_1 \xrightarrow{s_1}{}^{s} \cdots \xrightarrow{s_{m-1}}{}^{s} g_{m-1} \; {}^{t}\xleftarrow{t_{m-1}} f_m = r,$$

where $g_i$'s are source arrows into $x$, $s_i$'s are source arrows in $A \downarrow x$, and $t_i$'s are target arrows in $A \downarrow x$.

# Tree structures on pasting diagrams



The pasting diagram on the left has the tree structure on the right. Dots and lines in the tree correspond to 2-dimensional cells and 1-dimensional cells, respectively, in the pasting diagram.

# Opetopic set axioms

## Axiom (O7)

For every target arrow $f : y \to^t x$ in $A$ and object $z : A$ of degree $\leq \mathbf{deg}(y) - 2$, the postcomposition map $f_! : \mathbf{Arr}_A(z, y) \to \mathbf{Arr}_A(z, x)$ is injective.

Two ways to embed an $n$-cell to $(n + k)$-cell for $k \geq 2$ are not distinguished.

## Axiom (O8)

For every $k \geq 3$, every $k$-step arrow $y \to^k x$ in $A$ factors as $f \circ g$ such that $f$ is a $(k - 1)$-step arrow and $g$ is a 1-step arrow.

The 1-step arrows generate $A$.

# Opetopes

## Definition

An **opetope** is an opetopic set in which a terminal object exists.

Let $\mathbf{O}\mathcal{S}et$ denote the category of small opetopic sets whose morphisms are those functors preserving degrees, source arrows, and target arrows. Let $\mathbb{O} \subset \mathbf{O}\mathcal{S}et$ denote the full subcategory spanned by opetopes.

# Results

- $\mathbf{OSet} \simeq \mathbf{Psh}(\mathbb{O})$.
- Definition of pasting diagrams.
- Substitution and grafting of pasting diagrams.
- Equivalence with the polynomial monad definition by Kock, Joyal, Batanin, and Mascari [7].
- Presentation of the category of opetopes equivalent to Ho Thanh's [5].

# References I

[1] John C. Baez and James Dolan. "Higher-dimensional algebra. III. $n$-categories and the algebra of opetopes". In: *Adv. Math.* 135.2 (1998), pp. 145–206. DOI: 10.1006/aima.1997.1695.

[2] Clark Barwick and Christopher Schommer-Pries. "On the unicity of the theory of higher categories". In: *J. Amer. Math. Soc.* 34.4 (2021), pp. 1011–1058. DOI: 10.1090/jams/972.

[3] Pierre-Louis Curien, Cédric Ho Thanh, and Samuel Mimram. "Type theoretical approaches to opetopes". In: *Higher Structures* 6.1 (2022), pp. 80–181.

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[4] Claudio Hermida, Michael Makkai, and John Power. "On weak higher-dimensional categories. I. 3". In: *J. Pure Appl. Algebra* 166.1-2 (2002), pp. 83–104. DOI: 10.1016/S0022-4049(01)00014-7.

[5] Cédric Ho Thanh. *The equivalence between many-to-one polygraphs and opetopic sets*. 2021. arXiv: 1806.08645v4.

[6] Martin Hofmann and Thomas Streicher. "The groupoid interpretation of type theory". In: *Twenty-five years of constructive type theory (Venice, 1995)*. Vol. 36. Oxford Logic Guides. New York: Oxford Univ. Press, 1998, pp. 83–111. DOI: 10.1093/oso/9780198501275.003.0008.

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[7] Joachim Kock, André Joyal, Michael Batanin, and Jean-François Mascari. "Polynomial functors and opetopes". In: *Adv. Math.* 224.6 (2010), pp. 2690–2737. DOI: 10.1016/j.aim.2010.02.012.

[8] Tom Leinster. *Higher Operads, Higher Categories*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2004. DOI: 10.1017/CBO9780511525896. arXiv: math/0305049v1.

[9] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study, 2013. URL: http://homotopytypetheory.org/book/.

# Local rigidity

## Proposition

*Let* $F_1, F_2 : A \to A'$ *be morphisms of opetopic sets,* $x : A$*, and* $x' : A'$ *such that* $F_1(x) = F_2(x) = x'$*. Then* $F_1 \downarrow x, F_2 \downarrow x : A \downarrow x \to A' \downarrow x'$ *are identical.*

## Proposition

*Let* $F : A \to A'$ *be a morphism of opetopic sets and* $x : A$*. Then* $F \downarrow x : A \downarrow x \to A' \downarrow F(x)$ *is an equivalence.*

# Local rigidity

**Corollary**

$\mathbb{O}$ *is a gaunt category.*

**Corollary**

*Every morphism of opetopic sets is a discrete fibration.*

**Corollary**

$\mathbf{OSet} \downarrow A \simeq \mathbf{Psh}(A)$ *for every* $A : \mathbf{OSet}$.

# Local finiteness

## Proposition

*Let $A$ be an opetopic set. Then $A \downarrow x$ is finite for every $x : A$.*

## Corollary

*Every opetope is finite.*

## Corollary

$\mathbb{O}$ *is small.*

# The opetopic set of opetopes

We extend $\mathbb{O}$ to a preoeptopic set.

- $\mathbf{deg}_{\mathbb{O}}(A) \equiv \mathbf{deg}_A(*_A)$, where $*_A : A$ is the terminal object.
- $F : A' \to A$ is a source/target arrow if $F(*_{A'}) \to *_A$ is a source/target arrow.

## Proposition

*Let $A$ be an opetopic set. The morphism of preopetopic sets $A \to \mathbb{O} \downarrow A$ that sends $x : A$ to the forgetful functor $x_! : A \downarrow x \to A$ is an equivalence.*

## Corollary

$\mathbb{O}$ *is an opetopic set.*

# The terminal opetopic set

**Proposition**

$\mathbb{O} : \mathbf{OSet}$ *is the terminal object.*

**Proof.**

$(x \mapsto A \downarrow x) : A \to \mathbb{O}$ is the unique morphism. □

**Corollary**

$\mathbf{OSet} \simeq \mathbf{Psh}(\mathbb{O})$.

# The polynomial monad definition of opetopes

By Kock, Joyal, Batanin, and Mascari [7].

- A polynomial functor on $I$ is an endofunctor on $\mathbf{Set} \downarrow I$ of the form $P(X)_i = \coprod_{b:\mathbf{B}(P)_i} \prod_{e:\mathsf{E}_P(b)} X_{\mathsf{s}_P(e)}$.
- A polynomial monad on $I$ is a monad on $\mathbf{Set} \downarrow I$ whose underlying functor is a polynomial functor and unit and multiplication are cartesian natural transformations.
- For every polynomial monad $P$ on $I$, there is a polynomial monad $P^+$ on $\mathbf{B}(P)$, called the **Baez-Dolan construction**, such that $\mathbf{Alg}(P^+) \simeq \mathbf{PM}_I \downarrow P$.
- The set of KJBM $n$-opetopes $\mathbb{O}_n^{\mathrm{KJBM}}$ and the polynomial monad $\mathsf{Z}_n$ on $\mathbb{O}_n^{\mathrm{KJBM}}$ are defined by $\mathbb{O}_0^{\mathrm{KJBM}} \equiv 1$, $\mathsf{Z}_0(X) \equiv X$, $\mathbb{O}_{n+1}^{\mathrm{KJBM}} \equiv \mathbf{B}(\mathsf{Z}_n)$, and $\mathsf{Z}_{n+1} \equiv \mathsf{Z}_n^+$.

# Equivalence with the polynomial monad definition

## Theorem

$\mathbb{O}_n \simeq \mathbb{O}_n^{\mathrm{KJBM}}$

## Proof sketch.

Construct a polynomial monad $\mathbf{Y}_n$ on $\mathbb{O}_n$ and show that $\mathbf{Y}_0 \simeq \mathbf{Z}_0$ and $\mathbf{Y}_{n+1} \simeq \mathbf{Y}_n^+$. □

There are two compositional structures on pasting diagrams, **substitution** and **grafting**. The polynomial monad structure on $\mathbf{Y}_n$ is defined by substitution, and the equivalence $\mathbf{Y}_{n+1} \simeq \mathbf{Y}_n^+$ is proved by interaction between substitution and grafting.

# Categorical equivalence

Ho Thanh [5] gives a definition of the category of opetopes, whose objects are the KJBM opetopes, by generators and relations. Our category of opetopes $\mathbb{O}$ has the following presentation, which is shown equivalent to Ho Thanh's.

## Proposition

*Let $\mathrm{A}$ be an opetopic set. Then the underlying category of $\mathrm{A}$ is presented by:*

Generators *all the* 1*-step arrows in $\mathrm{A}$;*

Relations *all the equations $\mathrm{f}_1 \circ \mathrm{g}_1 = \mathrm{f}_2 \circ \mathrm{g}_2$ that hold in $\mathrm{A}$ such that $(\mathrm{f}_1, \mathrm{g}_1)$ is heterogeneous and $(\mathrm{f}_2, \mathrm{g}_2)$ is homogeneous.*