A universal property of the $(\infty, 1)$-category of presentable $(\infty, 1)$-categories

Taichi Uemura

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1 Introduction

In this note, we prove the following universal property of the $(\infty, 1)$-category of presentable $(n, 1)$-categories.

Theorem 1.1. *Let $-1 \leq n \leq \infty$. The $(\infty, 1)$-category of presentable $(n, 1)$-categories and right adjoint functors between them is the initial $(\infty, 1)$-category equipped with:*

- *small limits;*
- *an exponentiable morphism $\pi : \mathbf{S}_\bullet \to \mathbf{S}$*

satisfying that π is univalent and $(n-1)$-truncated and that π-small morphisms are closed under identities, composition, diagonals, and small products.

We also prove variants parameterized by accessibility rank.

Theorem 1.2. *Let $-1 \leq n \leq \infty$. For a regular cardinal λ, the $(\infty, 1)$-category of λ-presentable $(n, 1)$-categories and right adjoint functors between them preserving λ-filtered colimits is the initial $(\infty, 1)$-category equipped with:*

- *small limits;*
- *an exponentiable morphism $\pi : \mathbf{S}_\bullet \to \mathbf{S}$*

satisfying that π is univalent and $(n-1)$-truncated and that π-small morphisms are closed under identities, composition, diagonals, λ-small products, and retracts.

We prove the case when $n = \infty$ in Section 7. The case when $n < \infty$ is derived from it and proved in Section 8. In Section 3, we recall the definition of and basic facts about *exponentiable morphisms*. In Section 4, we recall the notion of *univalence* and the definition of *small morphisms* with respect to a given univalent morphism. We review the theory of *presentable $(\infty, 1)$-categories* in Section 5. One of the most important theorems is the *Gabriel-Ulmer duality*, which asserts that the $(\infty, 1)$-category of λ-presentable $(\infty, 1)$-categories is contravariantly equivalent to the $(\infty, 1)$-category of $(\infty, 1)$-categories with λ-small limits and splittings of idempotents. Our main theorem will follow from the structure of the latter $(\infty, 1)$-category. A key fact is that a slice of an $(\infty, 1)$-category with λ-small limits is a certain pushout in the $(\infty, 1)$-category of $(\infty, 1)$-categories with λ-small limits, which we review in Section 6.

Background and related work The main theorem is considered as a variant of the author's previous work [20] in the sense that both characterize $(\infty, 1)$-categories of *theories* in terms of *exponentiable morphisms*. The previous result characterizes the opposite of the category of generalized algebraic theories in terms of exponentiable morphisms. By the Gabriel-Ulmer duality, presentable categories are identified with limit theories, and presentable $(\infty, 1)$-categories are identified with "limit ∞-theories". Therefore, the main theorem in this note characterizes the $(\infty, 1)$-category of limit ∞-theories in terms of exponentiable morphisms.

The $(\infty, 1)$-category of limit ∞-theories itself can be regarded as a (large) limit ∞-theory, since it admits small limits. The exponentiability of π in the statement of the main theorem suggests that this limit ∞-theory is more natural to think of as a *second-order* limit ∞-theory in the sense that it contains operators from function types over fibers of π. This is analogous to the result of Fiore and Mahmoud [5, 4] that the algebraic theory of clones, which are equivalent to single-sorted algebraic theories, is simply the second-order algebraic theory of objects. Arkor and McDermott [1] observe its higher-order variant: the algebraic theory of k-th-order single-sorted algebraic theories is the $(k+1)$-th-order algebraic theory of objects. Our main theorem could be explained in a suitable framework of higher-order limit ∞-theories.

Hoang Kim Nguyen and the author gave in [13, Corollary 5.21] a universal property of similar kind for the opposite of the $(\infty, 1)$-category of $(\infty, 1)$-categories with finite limits: it is the initial $(\infty, 1)$-category with small limits equipped with a univalent exponentiable morphism u such that u-small morphisms are closed under identities, composition, and diagonals. This is different from Theorem 1.2 for $\lambda = \aleph_0$, because finitely presentable $(\infty, 1)$-categories are contravariantly equivalent to $(\infty, 1)$-categories with finite limits *and splittings of idempotents*. Nevertheless, a minor modification of the proof could give an alternative, direct proof of [13, Corollary 5.21] (the original proof uses the theory of ∞-type theories which are introduced for other purposes).

It is worth pointing out similarity between the main theorem and Kaposi and Kovács's type theory for defining higher inductive types [9]. Their type theory has a universe, corresponding to our $\pi : \mathbf{S}_\bullet \to \mathbf{S}$, and dependent function types over types from that universe, corresponding to the exponentiability of our π. This is not a coincidence, because higher inductive types are to be initial algebras [18] for limit ∞-theories. Note that the universe of Kaposi and Kovács's type theory is not univalent. Univalence is not necessary for presenting limit ∞-theories but gives a correct notion of equivalence between limit ∞-theories:

two limit ∞-theories are equivalent when the $(\infty, 1)$-categories of models are equivalent.

2 Preliminaries

By an ∞-*category* we mean a weak higher category in which all the m-cells for $m > 1$ are invertible. For concreteness, we use *quasi-categories* [11, 8, 3] as models of ∞-categories, but all the results are proved model-independently. For an ∞-category $\mathcal{C}$, let $\mathrm{Obj}(\mathcal{C})$ denote the space of objects, that is, the largest ∞-groupoid contained in $\mathcal{C}$. For objects $x, y \in \mathcal{C}$, let $\mathrm{Map}_{\mathcal{C}}(x, y)$ denote the mapping space from x to y and $\mathrm{Eq}_{\mathcal{C}}(x, y) \subset \mathrm{Map}_{\mathcal{C}}(x, y)$ the subspace of invertible morphisms.

By a *subcategory of* $\mathcal{C}$ we mean a monomorphism $F : \mathcal{C}' \to \mathcal{C}$ in the ∞-category of ∞-categories. This is equivalent to that F is faithful and, for any objects $x, y \in \mathcal{C}'$, the induced monomorphism $\mathrm{Eq}_{\mathcal{C}'}(x, y) \to \mathrm{Eq}_{\mathcal{C}}(F(x), F(y))$ is surjective. A subcategory of $\mathcal{C}$ is usually specified by a subclass $\mathcal{C}'_0$ of objects in $\mathcal{C}$ and a subclass $\mathcal{C}'_1$ of morphisms in $\mathcal{C}$ between objects in $\mathcal{C}'_0$ satisfying that $\mathcal{C}'_1$ is closed under identities and composition and that any equivalence in $\mathcal{C}$ between objects in $\mathcal{C}'_0$ belongs to $\mathcal{C}'_1$. A *full subcategory of* $\mathcal{C}$ is a fully faithful functor $F : \mathcal{C}' \to \mathcal{C}$. It is indeed a subcategory of $\mathcal{C}$ as $\mathrm{Eq}_{\mathcal{C}'}(x, y) \simeq \mathrm{Eq}_{\mathcal{C}}(F(x), F(y))$. A full subcategory of $\mathcal{C}$ is usually specified by a subclass of objects in $\mathcal{C}$.

λ denotes a regular cardinal. α and β denote inaccessible cardinals.

$\mathbf{S}(\alpha)$ denotes the ∞-category of α-small spaces. We may suppress (α) when size distinction is not important. $\mathbf{Cat}(\alpha)$ denotes the ∞-category of α-small ∞-categories. For ∞-categories $\mathcal{C}$ and $\mathcal{D}$, let $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ denote the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$ and natural transformations between them. When $\mathcal{C}$ and $\mathcal{D}$ have λ-small limits, we define $\mathbf{Fun}_{\mathrm{Lex}(\lambda)}(\mathcal{C}, \mathcal{D}) \subset \mathbf{Fun}(\mathcal{C}, \mathcal{D})$ to be the full subcategory spanned by functors preserving λ-small limits. We define $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\alpha) \subset \mathbf{Cat}(\alpha)$ to be the subcategory whose objects are the ∞-categories with λ-small limits *and splittings of idempotents* and whose morphisms are the functors between them preserving λ-small limits. Note that splittings of idempotents are redundant when $\lambda > \aleph_0$ since they are countable limits.

3 Exponentiable morphisms

We collect basic facts about exponentiable morphisms.

Notation 3.1. Let $u : y \to x$ be a morphism in an ∞-category $\mathcal{C}$ with finite limits. Let $u^* : \mathcal{C}_{/x} \to \mathcal{C}_{/y}$ denote the pullback functor along u, and let $u_!$ denote its left adjoint, that is, the postcomposition functor with u. When x is the terminal object, we write y^* and $y_!$ for u^* and $u_!$, respectively.

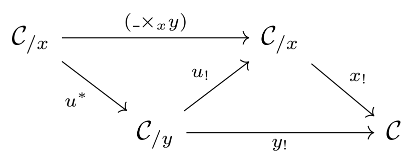
Definition 3.2. A morphism $u : y \to x$ in an ∞-category $\mathcal{C}$ with finite limits is *exponentiable* if the pullback functor $u^* : \mathcal{C}_{/x} \to \mathcal{C}_{/y}$ has a right adjoint u_* called the *pushforward along* u.

Proposition 3.3. *For a morphism $u : y \to x$ in an ∞-category $\mathcal{C}$ with finite limits, the following are equivalent:*

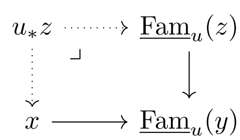
1. *u is exponentiable;*
2. *the functor $(\_ \times_x y) : \mathcal{C}_{/x} \to \mathcal{C}_{/x}$ has a right adjoint $\underline{\mathrm{Map}}_x(y, \_)$;*
3. *the composite $\mathcal{C}_{/x} \xrightarrow{u^*} \mathcal{C}_{/y} \xrightarrow{y_!} \mathcal{C}$ has a right adjoint $\underline{\mathrm{Fam}}_u$.*

Moreover, for any functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories with finite limits preserving finite limits and sending u to an exponentiable morphism, F commutes with one of u_*, $\underline{\mathrm{Map}}_x(y, \_)$, and $\underline{\mathrm{Fam}}_u$ if and only if F commutes with all of them.

Proof. This is well-known in the 1-categorical case [e.g. 14, Corollary 1.2]. We give a proof to confirm that this is also true in the ∞-context. We have the following commutative diagram.



Since $x_!$, $y_!$, and $u_!$ are left adjoints, $1 \Rightarrow 2$ and $2 \Rightarrow 3$ follow. Suppose that $\underline{\mathrm{Fam}}_u$ exists. Then, for any $z \in \mathcal{C}_{/y}$, the pushforward $u_* z \in \mathcal{C}_{/x}$ is defined by the pullback

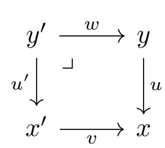


where the bottom morphism is the unit for $\underline{\mathrm{Fam}}_u$ at $\mathrm{id}_x \in \mathcal{C}_{/x}$. The last assertion is clear from the construction of u_*, $\underline{\mathrm{Map}}_x(y, \_)$, and $\underline{\mathrm{Fam}}_u$ from each other. □

Example 3.4. The forgetful functor $\pi(\alpha) : \mathbf{S}_\bullet(\alpha) \to \mathbf{S}(\alpha)$ from the ∞-category of α-small pointed spaces to the ∞-category of α-small spaces is exponentiable in $\mathbf{Cat}(\beta)$ where $\alpha < \beta$. Indeed, it is a left fibration and then [2, Corollary A.22] applies. More concretely, $\underline{\mathrm{Fam}}_{\pi(\alpha)}(\mathcal{C})$ for $\mathcal{C} \in \mathbf{Cat}(\beta)$ is the so-called *family fibration* and obtained by the Grothendieck construction for the functor

$$\mathbf{S}^{\mathrm{op}}(\alpha) \ni A \mapsto \mathcal{C}^A \in \mathbf{Cat}(\beta).$$

Proposition 3.5. *Let*



be a pullback in an ∞-category $\mathcal{C}$ with finite limits. If u is exponentiable, then so is u'. Moreover, for a functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories with finite limits preserving finite limits and sending u to an exponentiable morphism, if F commutes with u_*, then it commutes with u'_*.

Proof. This is also well-known in the 1-categorical case [e.g. 14, Corollary 1.4], and the proof also works in the ∞-context. Indeed, we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{C}_{/x'} & \xrightarrow{(u')^*} & \mathcal{C}_{/y'} & & \\ {\scriptstyle v_!}\downarrow & & \downarrow{\scriptstyle w_!} & \searrow^{y'_!} & \\ \mathcal{C}_{/x} & \xrightarrow[u^*]{} & \mathcal{C}_{/y} & \xrightarrow[y_!]{} & \mathcal{C} \end{array}$$

and $v_!$ has the right adjoint v^*, so $\underline{\mathrm{Fam}}_{u'}$ is constructed as $v^* \circ \underline{\mathrm{Fam}}_u$. The last assertion is clear from this construction of $\underline{\mathrm{Fam}}_{u'}$. □

4 Complete universes

We think of a morphism $u : y \to x$ in an ∞-category with finite limits as a *universe*. We introduce some concepts around universes.

We recall the notion of *univalence* [7, 16].

Definition 4.1. A morphism $u : y \to x$ in an ∞-category with finite limits $\mathcal{C}$ is *univalent* if, for any object $z \in \mathcal{C}$, the map of spaces

$$\mathrm{Map}_{\mathcal{C}}(z, x) \ni v \mapsto v^* y \in \mathrm{Obj}(\mathcal{C}_{/z})$$

is mono.

Example 4.2. $\pi(\alpha) : \mathbf{S}_\bullet(\alpha) \to \mathbf{S}(\alpha)$ is univalent in $\mathbf{Cat}(\beta)$. This is because it classifies left fibrations with α-small fibers in the sense that, for any $\mathcal{C} \in \mathbf{Cat}(\beta)$, the functor

$$\mathbf{Fun}(\mathcal{C}, \mathbf{S}(\alpha)) \ni F \mapsto F^* \mathbf{S}_\bullet(\alpha) \in \mathbf{Cat}(\beta)_{/\mathcal{C}}$$

is fully faithful and its image is the class of left fibrations over $\mathcal{C}$ with α-small fibers.

Proposition 4.3. *Let $\mathcal{C}$ be an ∞-category with finite limits and $\mathcal{C}' \subset \mathcal{C}$ a subcategory closed under finite limits. If a morphism $u : y \to x$ in $\mathcal{C}'$ is univalent in $\mathcal{C}$, then it is univalent in $\mathcal{C}'$.*

Proof. Let $z \in \mathcal{C}'$ be an object. Since $\mathcal{C}' \subset \mathcal{C}$ is closed under finite limits, we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}'}(z, x) & \xrightarrow{v \mapsto v^* y} & \mathrm{Obj}(\mathcal{C}'_{/z}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Map}_{\mathcal{C}}(z, x) & \xhookrightarrow[v \mapsto v^* y]{} & \mathrm{Obj}(\mathcal{C}_{/z}). \end{array}$$

The vertical maps are mono since $\mathcal{C}' \to \mathcal{C}$ is mono, and the bottom map is mono since u is univalent. Hence, the top map is mono as well. □

A univalent morphism induces an indexed ∞-category consisting of *small morphisms.*

Definition 4.4. Let $u : y \to x$ be a univalent morphism in an ∞-category $\mathcal{C}$ with finite limits. We say a morphism $u' : y' \to x'$ is *u-small* if there exists a pullback

$$\begin{array}{ccc} y' & \dashrightarrow & y \\ {\scriptstyle u'}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ x' & \dashrightarrow & x. \end{array}$$

Note that the univalence of u implies that such a pullback is unique.

Construction 4.5. For a univalent morphism $u : y \to x$ in an ∞-category $\mathcal{C}$ with finite limits, let $\mathbf{O}^u_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ be the functor sending $z \in \mathcal{C}$ to the full subcategory of $\mathcal{C}_{/z}$ spanned by the u-small morphisms. The action of morphisms is given by pullback. For a functor $F : \mathcal{C} \to \mathcal{D}$ preserving finite limits, we have a canonical natural transformation $\mathbf{O}^u_{\mathcal{C}}(z) \to \mathbf{O}^{F(u)}_{\mathcal{D}}(F(z))$ for $z \in \mathcal{C}$.

We consider when $\mathbf{O}^u_{\mathcal{C}}(z)$ has certain limits.

Definition 4.6. Let $\mathcal{C} \in \mathbf{Cat}_{\mathrm{Lex}(\lambda)}$. By a *$\lambda$-complete universe in $\mathcal{C}$* we mean a univalent exponentiable morphism u in $\mathcal{C}$ such that u-small morphisms are closed under identities, composition, diagonals, λ-small products, and retracts.

Remark 4.7. Closure under retracts is redundant when $\lambda > \aleph_0$.

Remark 4.8. Closure under λ-small products is redundant when $\lambda = \aleph_0$.

Example 4.9. $\pi(\alpha) : \mathbf{S}_\bullet(\alpha) \to \mathbf{S}(\alpha)$ is an α-complete universe in $\mathbf{Cat}(\beta)$, since left fibrations with α-small fibers are closed under identities, composition, diagonals, α-small products, and retracts.

Proposition 4.10. *Let $u : y \to x$ be a λ-complete universe in $\mathcal{C} \in \mathbf{Cat}_{\mathrm{Lex}(\lambda)}$. The functor $\mathbf{O}^u_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ factors through $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}$.*

Proof. By the closure properties of u-small morphisms. □

Proposition 4.11. *Let $u : y \to x$ be a λ-complete universe in $\mathcal{C} \in \mathbf{Cat}_{\mathrm{Lex}(\lambda)}$. For morphisms $v : y' \to x'$ and $w : z' \to y'$, if v is u-small, then w is u-small if and only if $v \circ w$ is.*

Proof. Because u-small morphisms are closed under pullbacks, composition, and diagonals. □

$\mathbf{O}^u_{\mathcal{C}}$ is representable in the following sense.

Proposition 4.12. *Let $u : y \to x$ be a λ-complete universe in $\mathcal{C} \in \mathbf{Cat}_{\mathrm{Lex}(\lambda)}$.*

1. $\mathrm{Obj}(\mathbf{O}^u_{\mathcal{C}}(z)) \simeq \mathrm{Map}_{\mathcal{C}}(z, x)$;
2. $\mathrm{Obj}(\mathbf{O}^u_{\mathcal{C}}(z)_{\mathrm{id}_z/}) \simeq \mathrm{Map}_{\mathcal{C}}(z, y)$;
3. *The presheaf* $z \mapsto \mathrm{Obj}(\mathbf{O}^u_{\mathcal{C}}(z)^{\to})$ *is representable. The representing object is denoted by* $\underline{\mathrm{Map}}(u)$.

Proof. The first and second equivalences are immediate by construction. For the last, we define $\underline{\mathrm{Map}}(u) \in \mathcal{C}_{/x\times x}$ to be $\underline{\mathrm{Map}}_{x\times x}(y \times x, x \times y)$. Note that, by Proposition 3.5, the morphism $u \times x : y \times x \to x \times x$ is exponentiable. □

Remark 4.13. $\underline{\mathrm{Map}}(u)$ is part of the complete Segal object associated to u [15].

We introduce an ∞-category of ∞-categories equipped with a λ-complete universe.

Definition 4.14. For $\lambda \leq \alpha < \beta$, we define $\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$ to be the ∞-category whose objects are the β-small ∞-categories with α-small limits and equipped with a λ-complete universe and whose morphisms are the functors between them preserving α-small limits, specified λ-complete universes, and pushforwards along specified λ-complete universes.

5 Presentable ∞-categories

We recall the definition and basic properties of presentable ∞-categories [11, Section 5.5]. We also find a λ-complete universe in an ∞-category of presentable ∞-categories (Proposition 5.3). The *Gabriel-Ulmer duality* is rephrased using that universe (Proposition 5.8).

Definition 5.1. For $\lambda < \alpha$, we say an ∞-category $\mathcal{X}$ is (α, λ)*-presentable* if it is equivalent to $\mathbf{Ind}^{\alpha}_{\lambda}(\mathcal{C})$ for some α-small ∞-category $\mathcal{C}$ with λ-small colimits, where $\mathbf{Ind}^{\alpha}_{\lambda}$ is the completion under α-small λ-filtered colimits. For convention, we say $\mathcal{X}$ is (α, α)*-presentable* if it is (α, λ)-presentable for some $\lambda < \alpha$. By definition, any (α, λ)-presentable ∞-category is β-small for $\lambda \leq \alpha < \beta$. We define $\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha) \subset \mathbf{Cat}(\beta)$ to be the subcategory spanned by the (α, λ)-presentable ∞-categories and right adjoint functors between them preserving α-small λ-filtered colimits.

Proposition 5.2 ([11, Proposition 5.5.7.6])**.** $\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha) \subset \mathbf{Cat}(\beta)$ *is closed under* α*-small limits for* $\lambda \leq \alpha < \beta$.

Proposition 5.3. *For* $\lambda \leq \alpha$*, the functor* $\pi(\alpha) : \mathbf{S}_{\bullet}(\alpha) \to \mathbf{S}(\alpha)$ *belongs to* $\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha)$ *and is a* λ*-complete universe.*

Proposition 5.3 is split into a few lemmas.

Lemma 5.4. $\pi(\alpha)$ *belongs to* $\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha)$ *and is univalent.*

Proof. Since $\mathbf{S}_\bullet(\alpha) \simeq \mathbf{S}(\alpha)_{1/}$, it belongs to $\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)$ because (α, λ)-presentable ∞-categories are closed under coslice [11, Proposition 5.5.3.11]. The projection $\pi(\alpha)$ is in $\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)$ as it has the left adjoint $(\_ + 1)$ and preserves filtered colimits. The univalence of $\pi(\alpha)$ follows from Propositions 4.3 and 5.2 and Example 4.2. □

Lemma 5.5. *For a morphism $p : \mathcal{Y} \to \mathcal{X}$ in $\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)$, the following are equivalent:*

1. *p is $\pi(\alpha)$-small in $\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)$;*
2. *the functor p is a left fibration representable by some λ-compact object of $\mathcal{X}$;*
3. *the functor p is a left fibration and preserves λ-compact objects.*

Proof. 1 ⇒ 2. Suppose that p is the pullback of $\pi(\alpha)$ along a morphism $F : \mathcal{X} \to \mathbf{S}(\alpha)$. By definition, F has a left adjoint and preserves α-small λ-filtered colimits, and thus F is representable by some λ-compact object of $\mathcal{X}$.

2 ⇒ 1. If p is representable by a λ-compact object $A \in \mathcal{X}$, then it is the pullback of $\pi(\alpha)$ along the functor $\mathrm{Map}_{\mathcal{X}}(A, \_) : \mathcal{X} \to \mathbf{S}(\alpha)$, which belongs to $\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)$.

2 ⇒ 3. The left fibration $\mathrm{cod} : \mathcal{X}_{A/} \to \mathcal{X}$ preserves λ-compact objects whenever $A \in \mathcal{X}$ is λ-compact.

3 ⇒ 2. If p is a left fibration, then it is representable because $\mathcal{Y}$ has an initial object. If $\mathrm{cod} : \mathcal{X}_{A/} \to \mathcal{X}$ preserves λ-compact objects, then A is λ-compact as it is the image of the initial object by cod. □

Lemma 5.6. *$\pi(\alpha)$-small morphisms in $\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)$ are closed under identities, composition, diagonals, λ-small products, and retracts.*

Proof. The closure under identities and composition is immediate from Lemma 5.5. For diagonals, it suffices to show that $\mathcal{X}_{A/} \to \mathcal{X}_{A/} \times_{\mathcal{X}} \mathcal{X}_{A/} \simeq \mathcal{X}_{A+A/}$ is $\pi(\alpha)$-small whenever $A \in \mathcal{X}$ is λ-compact, but this is true because the codiagonal $A + A \to A$ is a λ-compact object of $\mathcal{X}_{A+A/}$. For λ-small products (retracts), use the fact that λ-compact objects are closed under λ-small coproducts (retracts). □

Proof of Proposition 5.3. It remains to show that $\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha) \subset \mathbf{Cat}(\beta)$ is closed under $\underline{\mathrm{Fam}}_{\pi(\alpha)}$. For a (α, λ)-presentable ∞-category $\mathcal{X}$, the cartesian fibration $\underline{\mathrm{Fam}}_{\pi(\alpha)}(\mathcal{X}) \to \mathbf{S}(\alpha)$ is a presentable fibration in the sense of Gepner, Haugseng, and Nikolaus [6]. It then follows that $\underline{\mathrm{Fam}}_{\pi(\alpha)}(\mathcal{X}) \to \mathbf{S}(\alpha)$ belongs to $\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)$ [6, Theorem 10.3]. Note that [6, Theorem 10.3] does not mention the accessibility rank, but we can calculate it from the proof of that theorem. It is straightforward to see that the unit and counit for $\underline{\mathrm{Fam}}_{\pi(\alpha)}$ belong to $\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)$, and thus $\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha) \subset \mathbf{Cat}(\beta)$ is closed under $\underline{\mathrm{Fam}}_{\pi(\alpha)}$. □

By Propositions 5.2 and 5.3, $(\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha), \pi(\alpha))$ is an object of $\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$. We state a version of the *Gabriel-Ulmer duality* (Proposition 5.8).

Lemma 5.7. $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)$ *is* (β,α)*-presentable for* $\lambda \le \alpha < \beta$*.*

Proof. See [10, Lemma 4.8.4.2]. One can also construct $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)$ directly in $\mathbf{Pr}^{\mathrm{R}}_{\alpha}(\beta)$ using Proposition 5.2. □

Proposition 5.8. *For* $\lambda \le \alpha < \beta$*, the functor*

$$\mathbf{O}^{\pi(\alpha)}_{\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha)} : \mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha)^{\mathrm{op}} \to \mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)$$

is fully faithful, and its image is the class of α*-compact objects. Moreover, the inverse of the induced equivalence takes a* α*-compact object* $\mathcal{C} \in \mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)$ *to* $\mathbf{Fun}_{\mathrm{Lex}(\lambda)}(\mathcal{C}, \mathbf{S}(\alpha))$*.*

Proof. By Lemma 5.5, $\mathbf{O}^{\pi(\alpha)}_{\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha)}(\mathcal{X})$ is equivalent to the opposite of the full subcategory of $\mathcal{X}$ spanned by the λ-compact objects. The claim then follows from [11, Proposition 5.5.7.10] when $\lambda < \alpha$ and from [19, Proposition 5.1.4] when $\lambda = \alpha$. Note that in [11, Proposition 5.5.7.10], the inverse is given by $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\alpha) \ni \mathcal{C} \mapsto \mathbf{Ind}^{\alpha}_{\lambda}(\mathcal{C}^{\mathrm{op}}) \in \mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha)^{\mathrm{op}}$, but this is equivalent to $\mathbf{Fun}_{\mathrm{Lex}(\lambda)}(\_, \mathbf{S}(\alpha))$ by [11, Corollary 5.3.5.4], so the last assertion follows when $\lambda < \alpha$. When $\lambda = \alpha$, the last assertion is because $\mathcal{C} \simeq \mathbf{Fun}_{\mathrm{Lex}(\alpha)}(\mathcal{C}^{\mathrm{op}}, \mathbf{S}(\alpha))$ when $\mathcal{C}$ is (α,α)-presentable by [11, Proposition 5.5.2.2]. □

6 Slices of complete ∞-categories

Our main theorem is the initiality of $(\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha), \pi(\alpha))$ in $\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$ (Theorem 7.1). By Proposition 5.8, $\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha)$ is embedded into $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)^{\mathrm{op}}$, so it is good to study $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)$. The most important tool is the following universal property of slices.

Lemma 6.1 ([13, Proposition 3.25])**.** *Let* $\mathcal{C}$ *be an* ∞*-category with finite limits and* $x \in \mathcal{C}$ *an object. For any* ∞*-category* $\mathcal{D}$ *with finite limits, the square*

$$\begin{array}{ccc}
\mathrm{Obj}(\mathbf{Fun}_{\mathrm{Lex}(\aleph_0)}(\mathcal{C}_{/x}, \mathcal{D})) & \xrightarrow{\mathrm{ev}_{\Delta_x}} & \mathrm{Obj}(\mathcal{D}_{1/}) \\
{\scriptstyle (_\circ x^*)}\downarrow & & \downarrow{\scriptstyle \mathrm{cod}} \\
\mathrm{Obj}(\mathbf{Fun}_{\mathrm{Lex}(\aleph_0)}(\mathcal{C}, \mathcal{D})) & \xrightarrow[\mathrm{ev}_x]{} & \mathrm{Obj}(\mathcal{D})
\end{array}$$

is a pullback, where we regard the diagonal $\Delta_x : x \to x \times x$ *as a global section* $1 \to x^*x$ *in* $\mathcal{C}_{/x}$*. Moreover, the inverse of the induced map* $\mathrm{Obj}(\mathbf{Fun}_{\mathrm{Lex}(\aleph_0)}(\mathcal{C}_{/x}, \mathcal{D})) \to \mathrm{Obj}(\mathrm{Fun}_{\mathrm{Lex}(\aleph_0)}(\mathcal{C}, \mathcal{D})) \times_{\mathrm{Obj}(\mathcal{D})} \mathrm{Obj}(\mathcal{D}_{1/})$ *sends an object* (F, u) *to the composite*

$$\mathcal{C}_{/x} \xrightarrow{F_{/x}} \mathcal{D}_{/F(x)} \xrightarrow{u^*} \mathcal{D}_{/1} \simeq \mathcal{D}.$$

Proposition 6.2. *Let $\mathcal{C}$ be an ∞-category with λ-small limits and $x \in \mathcal{C}$ an object. For any ∞-category $\mathcal{D}$ with λ-small limits, the square*

$$\begin{array}{ccc} \mathrm{Obj}(\mathbf{Fun}_{\mathrm{Lex}(\lambda)}(\mathcal{C}_{/x}, \mathcal{D})) & \xrightarrow{\mathrm{ev}_{\Delta_x}} & \mathrm{Obj}(\mathcal{D}_{1/}) \\ {\scriptstyle (\_\circ x^*)}\downarrow & & \downarrow{\scriptstyle \mathrm{cod}} \\ \mathrm{Obj}(\mathbf{Fun}_{\mathrm{Lex}(\lambda)}(\mathcal{C}, \mathcal{D})) & \xrightarrow[\mathrm{ev}_x]{} & \mathrm{Obj}(\mathcal{D}) \end{array}$$

is a pullback.

Proof. The equivalence $\mathrm{Obj}(\mathbf{Fun}_{\mathrm{Lex}(\aleph_0)}(\mathcal{C}_{/x}, \mathcal{D})) \simeq \mathrm{Obj}(\mathbf{Fun}_{\mathrm{Lex}(\aleph_0)}(\mathcal{C}, \mathcal{D})) \times_{\mathrm{Obj}(\mathcal{D})} \mathrm{Obj}(\mathcal{D}_{1/})$ of Lemma 6.1 is restricted to an equivalence $\mathrm{Obj}(\mathbf{Fun}_{\mathrm{Lex}(\lambda)}(\mathcal{C}_{/x}, \mathcal{D})) \simeq \mathrm{Obj}(\mathbf{Fun}_{\mathrm{Lex}(\lambda)}(\mathcal{C}, \mathcal{D})) \times_{\mathrm{Obj}(\mathcal{D})} \mathrm{Obj}(\mathcal{D}_{1/})$, because $u^* \circ F_{/x}$ preserves λ-small limits whenever $F : \mathcal{C} \to \mathcal{D}$ does. □

Construction 6.3. We define $\langle \mathfrak{x} \rangle_\lambda$ to be the ∞-category with λ-small limits freely generated by one object $\mathfrak{x}$ and $\langle \mathfrak{s} : 1 \to \mathfrak{x} \rangle_\lambda$ the extension of $\langle \mathfrak{x} \rangle_\lambda$ by a global section $\mathfrak{s} : 1 \to \mathfrak{x}$. Let $\iota : \langle \mathfrak{x} \rangle_\lambda \to \langle \mathfrak{s} : 1 \to \mathfrak{x} \rangle_\lambda$ denote the canonical morphism in $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}$.

Corollary 6.4. *Let $\mathcal{C}$ be an ∞-category with λ-small limits and $x \in \mathcal{C}$ an object. We regard x and Δ_x as morphisms $\langle \mathfrak{x} \rangle_\lambda \to \mathcal{C}$ and $\langle \mathfrak{s} : 1 \to \mathfrak{x} \rangle_\lambda \to \mathcal{C}_{/x}$, respectively, in $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}$. Then*

$$\begin{array}{ccc} \langle \mathfrak{x} \rangle_\lambda & \xrightarrow{x} & \mathcal{C} \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle x^*} \\ \langle \mathfrak{s} : 1 \to \mathfrak{x} \rangle_\lambda & \xrightarrow[\Delta_x]{} & \mathcal{C}_{/x} \end{array}$$

is a pushout in $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}$. □

Corollary 6.5. *For $\lambda < \alpha$, the pushout functor*

$$\iota_! : \mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\alpha)_{\langle \mathfrak{x} \rangle_\lambda /} \to \mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\alpha)_{\langle \mathfrak{s}:1 \to \mathfrak{x} \rangle_\lambda /}$$

along ι preserves α-small limits. □

Under the Gabriel-Ulmer duality (Proposition 5.8), the morphism $\iota : \langle \mathfrak{x} \rangle_\lambda \to \langle \mathfrak{s} : 1 \to \mathfrak{x} \rangle_\lambda$ in $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)$ corresponds to the morphism $\pi(\alpha) : \mathbf{S}_\bullet(\alpha) \to \mathbf{S}(\alpha)$ in $\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)$.

Proposition 6.6. *The inclusion $\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha) \subset \mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)^{\mathrm{op}}$ takes pushforwards along $\pi(\alpha)$ to those along ι.*

Proof. By Corollary 6.5, this follows from Lemma 5.7 and Lemma 6.7 below. □

Lemma 6.7. *Let $u : A \to B$ be a morphism in a (α, λ)-presentable ∞-category $\mathcal{X}$. The morphism u is exponentiable in $\mathcal{X}^{\mathrm{op}}$ if and only if the pushout functor $u_! : \mathcal{X}_{A/} \to \mathcal{X}_{B/}$ preserves α-small limits. If this is the case, then $u_* : (\mathcal{X}^{\mathrm{op}})_{/B} \to (\mathcal{X}^{\mathrm{op}})_{/A}$ takes λ-compact objects in $\mathcal{X}_{B/}$ to λ-compact objects in $\mathcal{X}_{A/}$.*

Proof. Since $u_!$ preserves all colimits, this follows from the adjoint functor theorem [11, Corollary 5.5.2.9 (2)]. □

7 The main theorem

This section is devoted to the proof of the following theorem.

Theorem 7.1. $(\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha), \pi(\alpha))$ *is the initial object of* $\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$ *for* $\lambda \leq \alpha < \beta$.

The following criterion is useful.

Proposition 7.2 ([12, Proposition 2.2.2])**.** *If an* ∞*-category* $\mathcal{C}$ *has finite limits, then an object of* $\mathcal{C}$ *is initial if and only if its is initial in the homotopy category of* $\mathcal{C}$.

$\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$ has finite limits computed component-wise, and thus for Theorem 7.1 it is enough to construct for every $(\mathcal{C}, u) \in \mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$:

1. a morphism $!_{(\mathcal{C},u)} : (\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha), \pi(\alpha)) \to (\mathcal{C}, u)$;
2. an equivalence $!_{(\mathcal{C},u)} \simeq F$ for any morphism $F : (\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha), \pi(\alpha)) \to (\mathcal{C}, u)$.

For the existence of a morphism $!_{(\mathcal{C},u)} : (\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha), \pi(\alpha)) \to (\mathcal{C}, u)$, we construct a functor $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)^{\mathrm{op}} \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S}(\beta))$ (Construction 7.3) and show that its restriction to $\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha) \subset \mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)^{\mathrm{op}}$ (Proposition 5.8) factors through the Yoneda embedding (Lemma 7.6).

Construction 7.3. Given an object $(\mathcal{C}, u) \in \mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$, we define

$$\mathrm{N}_{\mathbf{O}^u_{\mathcal{C}}} : \mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)^{\mathrm{op}} \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S}(\beta))$$

to be the nerve of $\mathbf{O}^u_{\mathcal{C}} : \mathcal{C} \to \mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)^{\mathrm{op}}$, that is,

$$\mathrm{N}_{\mathbf{O}^u_{\mathcal{C}}}(\mathcal{D}) = \mathrm{Map}_{\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)}(\mathcal{D}, \mathbf{O}^u_{\mathcal{C}}(\_)).$$

For a morphism $F : (\mathcal{C}, u) \to (\mathcal{D}, v)$ in $\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\alpha)$, the natural transformation $\mathbf{O}^u_{\mathcal{C}} \Rightarrow \mathbf{O}^v_{\mathcal{D}} \circ F$ induces a natural transformation $\mathrm{N}_{\mathbf{O}^u_{\mathcal{C}}} \Rightarrow F^* \circ \mathrm{N}_{\mathbf{O}^v_{\mathcal{D}}} : \mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)^{\mathrm{op}} \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S}(\beta))$. By the adjunction $F_! \dashv F^*$, we have a natural transformation

$$F_! \circ \mathrm{N}_{\mathbf{O}^u_{\mathcal{C}}} \Rightarrow \mathrm{N}_{\mathbf{O}^v_{\mathcal{D}}} : \mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)^{\mathrm{op}} \to \mathbf{Fun}(\mathcal{D}^{\mathrm{op}}, \mathbf{S}(\beta)). \tag{1}$$

Construction 7.4. Let $\langle \mathfrak{s} : \mathfrak{x} \to \mathfrak{y} \rangle_{\lambda}$ denote the ∞-category with λ-small limits freely generated by one morphism $\mathfrak{s} : \mathfrak{x} \to \mathfrak{y}$.

Lemma 7.5. $\langle \mathfrak{x} \rangle_{\lambda}$ *and* $\langle \mathfrak{s} : \mathfrak{x} \to \mathfrak{y} \rangle_{\lambda}$ *form a strong generator for* $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)$. □

Lemma 7.6. *For any* $(\mathcal{C}, u) \in \mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$*, the restriction of* $\mathrm{N}_{\mathbf{O}^u_{\mathcal{C}}} : \mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)^{\mathrm{op}} \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S}(\beta))$ *to the* α*-compact objects factors through the Yoneda embedding* $\mathcal{C} \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S}(\beta))$*.*

Proof. By Lemma 7.5, it suffices to show the representability of $\mathrm{N}_{\mathbf{O}^u_{\mathcal{C}}}$ for $\langle \mathfrak{x} \rangle_\lambda$ and $\langle \mathfrak{s} : \mathfrak{x} \to \mathfrak{y} \rangle_\lambda$, but this is immediate from Proposition 4.12. □

Construction 7.7. Recall from Proposition 5.8 that $\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)^{\mathrm{op}}$ is the full subcategory of $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)$ spanned by the α-compact objects. Let $!_{\mathcal{C}} : \mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha) \to \mathcal{C}$ be the functor induced by Lemma 7.6.

Lemma 7.8. *For any* $(\mathcal{C}, u) \in \mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$*, the functor* $!_{\mathcal{C}} : \mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha) \to \mathcal{C}$ *is a morphism in* $\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$*.*

Proof. By definition, $!_{\mathcal{C}}$ preserves small limits. $!_{\mathcal{C}}$ sends $\pi(\alpha) : \mathbf{S}_\bullet(\alpha) \to \mathbf{S}(\alpha)$ to $u : y \to x$ by Proposition 4.12, because $\pi(\alpha)$ corresponds to the morphism $\iota : \langle \mathfrak{x} \rangle_\lambda \to \langle \mathfrak{s} : 1 \to \mathfrak{x} \rangle_\lambda$ in $\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)$ via the Gabriel-Ulmer duality (Proposition 5.8). For the preservation of the pushforward along $\pi(\alpha)$, let $\mathcal{X} \in \mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)$ be an object and take the corresponding α-compact object $\mathcal{D} \in \mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)$. We have

$$
\begin{aligned}
& \mathrm{Map}_{\mathcal{C}}(x', !_{\mathcal{C}}(\underline{\mathrm{Fam}}_{\pi(\alpha)}(\mathcal{X}))) \\
\simeq \quad & \{\text{definition}\} \\
& \mathrm{Map}_{\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)^{\mathrm{op}}}(\mathbf{O}^u_{\mathcal{C}}(x'), \underline{\mathrm{Fam}}_\iota(\mathcal{D})) \\
\simeq \quad & \{\text{Corollary 6.4}\} \\
& \sum_{y' \in \mathrm{Obj}(\mathbf{O}^u_{\mathcal{C}}(x'))} \mathrm{Map}_{\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)}(\mathcal{D}, \mathbf{O}^u_{\mathcal{C}}(x')_{/y'}) \\
\simeq \quad & \{\text{Proposition 4.11}\} \\
& \sum_{y' \in \mathrm{Obj}(\mathbf{O}^u_{\mathcal{C}}(x'))} \mathrm{Map}_{\mathbf{Cat}_{\mathrm{Lex}(\lambda)}(\beta)}(\mathcal{D}, \mathbf{O}^u_{\mathcal{C}}(y')) \\
\simeq \quad & \{\text{definition}\} \\
& \sum_{v \in \mathrm{Map}_{\mathcal{C}}(x', x)} \mathrm{Map}_{\mathcal{C}}(x' \times_x y, !_{\mathcal{C}}(\mathcal{X}))
\end{aligned}
$$

for any $x' \in \mathcal{C}$, and thus $!_{\mathcal{C}}(\underline{\mathrm{Fam}}_{\pi(\alpha)}(\mathcal{X})) \simeq \underline{\mathrm{Fam}}_u(!_{\mathcal{C}}(\mathcal{X}))$. □

We have seen the existence of $!_{\mathcal{C}} : (\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha), \pi(\alpha)) \to (\mathcal{C}, u)$. The uniqueness will follow from the functoriality of the construction of $!_{\mathcal{C}}$ (Lemmas 7.9 and 7.10).

Lemma 7.9. $!_{\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)}$ *is the identity.*

Proof. By construction. □

Lemma 7.10. *For any morphism* $F : (\mathcal{C}, u) \to (\mathcal{D}, v)$ *in* $\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$*, we have an equivalence*

$$F \circ !_{\mathcal{C}} \simeq !_{\mathcal{D}}.$$

Proof. It is enough to show that the canonical natural transformation $F_! \circ \mathrm{N}_{\mathbf{O}^u_{\mathcal{C}}} \Rightarrow \mathrm{N}_{\mathbf{O}^v_{\mathcal{D}}}$ (Eq. (1)) is invertible at α-compact objects. Since $F_!$, $\mathrm{N}_{\mathbf{O}^u_{\mathcal{C}}}$, and $\mathrm{N}_{\mathbf{O}^v_{\mathcal{D}}}$ preserve α-small limits, it suffices to show the invertibility at $\langle \mathfrak{x} \rangle_\lambda$ and $\langle \mathfrak{s} : \mathfrak{x} \to \mathfrak{y} \rangle_\lambda$ by Lemma 7.5, but this is immediate from the construction. ☐

For any morphism $F : (\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha), \pi(\alpha)) \to (\mathcal{C}, u)$ in $\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$, we have an equivalence $!_{\mathcal{C}} \simeq F$ as

$$
\begin{aligned}
& !_{\mathcal{C}} \\
\simeq \quad & \{\text{Lemma 7.10}\} \\
& F \circ !_{\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)} \\
\simeq \quad & \{\text{Lemma 7.9}\} \\
& F.
\end{aligned}
$$

This completes the proof of Theorem 7.1.

8 Presentable n-categories

We consider an n-categorical version of Theorem 7.1. For $-1 \le n \le \infty$, by an n-category we mean an ∞-category whose mapping spaces are $(n-1)$-truncated. For example, 1-categories are ordinary categories, 0-categories are posets, and (-1)-categories are subsingletons.

Construction 8.1. For $-1 \le n < \infty$, let $n\text{-}\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha) \subset \mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)$ denote the full subcategory spanned by (α, λ)-presentable n-categories. The inclusion has a coreflection, which takes $\mathcal{X} \in \mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)$ to the full subcategory $\mathcal{X}_{\le n-1} \subset \mathcal{X}$ spanned by the $(n-1)$-truncated objects.

Theorem 8.2. *$(n\text{-}\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha), \pi(\alpha)_{\le n-1})$ is the initial object in the full subcategory of $\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$ spanned by those objects $(\mathcal{C}, u)$ such that u is $(n-1)$-truncated, for $-1 \le n < \infty$ and $\lambda \le \alpha < \beta$.*

Remark 8.3. If a univalent morphism u is k-truncated for $k < \infty$, then u-small morphisms are closed under retracts whenever they are closed under identities, composition, and diagonals. This is because splittings of idempotents on k-truncated objects are finite limits.

Example 8.4. Consider the case when $n = 0$ and $\lambda = \aleph_0$. A univalent exponentiable monomorphism u is a $\aleph_0$-complete universe if and only if u-small morphisms are closed under identities and composition, because the diagonal of a monomorphism is an equivalence. That is, u is a dominance [17]. By the Gabriel-Ulmer duality (Proposition 5.8), $0\text{-}\mathbf{Pr}^{\mathrm{R}}_{\aleph_0}(\alpha)$ is equivalent to $\mathbf{Lat}_\wedge(\alpha)^{\mathrm{op}}$, the opposite of the ∞-category of α-small meet semilattices. Therefore, $\mathbf{Lat}_\wedge(\alpha)^{\mathrm{op}}$ is the initial ∞-category with α-small limits equipped with an exponentiable dominance.

In the rest of this section, we prove Theorem 8.2. We first recall a characterization of truncated objects and morphisms.

Construction 8.5. Let $\mathcal{C}$ be an ∞-category with finite limits. We define a functor $\Delta : \mathcal{C}^{\to} \to \mathcal{C}^{\to}$ by $(y \to x) \mapsto (y \to y \times_x y)$. Let $\Delta^k : \mathcal{C}^{\to} \to \mathcal{C}^{\to}$ denote the k-fold composition of Δ and let Δ_1^k be the composite $\mathcal{C} \xrightarrow{x \mapsto (x \to 1)} \mathcal{C}^{\to} \xrightarrow{\Delta^k} \mathcal{C}^{\to}$.

Proposition 8.6. *Let $\mathcal{C}$ be an ∞-category with finite limits and $k \geq -2$.*

1. *A morphism u in $\mathcal{C}$ is k-truncated if and only if $\Delta^{k+2}(u)$ is an equivalence.*
2. *An object x in $\mathcal{C}$ is k-truncated if and only if $\Delta_1^{k+2}(x)$ is an equivalence.*

Proof. This follows from [11, Lemma 5.5.6.15]. □

Given a λ-complete universe, we construct a subuniverse of k-truncated objects.

Lemma 8.7. *Let $u : y \to x$ be a univalent exponentiable morphism in an ∞-category $\mathcal{C}$ with finite limits. The morphism $\underline{\mathrm{id}}_u : x \to \underline{\mathrm{Map}}(u)$ corresponding to the identity on y is mono.*

Proof. Because $\underline{\mathrm{id}}_u$ represents the monomorphism $\mathrm{Obj}(\mathbf{O}^u_{\mathcal{C}}(x')) \ni y' \mapsto \mathrm{id}_{y'} \in \mathrm{Obj}(\mathbf{O}^u_{\mathcal{C}}(x')^{\to})$. □

Proposition 8.8. *Let $(\mathcal{C}, u : y \to x) \in \mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$ and $k \leq -2$. There exists a (necessarily) unique λ-complete universe $u_{\leq k} : y_{\leq k} \to x_{\leq k}$ in $\mathcal{C}$ such that the $u_{\leq k}$-small morphisms are precisely the k-truncated u-small morphisms. Moreover, if $F : (\mathcal{C}, u) \to (\mathcal{D}, v)$ is a morphism in $\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$, then F takes $u_{\leq k}$ to $v_{\leq k}$.*

Proof. Since the presheaf $z \mapsto \mathrm{Obj}(\mathbf{O}^u_{\mathcal{C}}(z)^{\to})$ is representable by $\underline{\mathrm{Map}}(u)$, we have a morphism $\underline{\Delta}_1^k : x \to \underline{\mathrm{Map}}(u)$ representing Δ_1^k. Take the pullback

$$\begin{array}{ccc} x_{\leq k} & \dashrightarrow & x \\ \Big\downarrow & \lrcorner & \Big\downarrow \underline{\mathrm{id}}_u \\ x & \xrightarrow[\underline{\Delta}_1^{k+2}]{} & \underline{\mathrm{Map}}(u), \end{array}$$

where $\underline{\mathrm{id}}_u$ is mono by Lemma 8.7. We define $u_{\leq k} : y_{\leq k} \to x_{\leq k}$ to be the pullback of u along the monomorphism $x_{\leq k} \to x$. Then $u_{\leq k}$ is univalent, and the $u_{\leq k}$-small morphisms are precisely the k-truncated u-small morphisms by Proposition 8.6. $u_{\leq k}$ is a λ-complete universe because k-truncated morphisms are closed under identities, composition, diagonals, λ-small products, and retracts. The last assertion follows from the construction of $u_{\leq k}$. □

Example 8.9. The λ-complete universe $\pi(\alpha)_{\leq n-1} : \mathbf{S}_\bullet(\alpha)_{\leq n-1} \to \mathbf{S}(\alpha)_{\leq n-1}$ in $\mathbf{Pr}^{\mathrm{R}}_\lambda(\alpha)$ obtained by Proposition 8.8 coincides with the coreflection described in Construction 8.1.

Lemma 8.10. *The inclusion* $n\text{-}\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha) \to \mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha)$ *defines a morphism*

$$(n\text{-}\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha), \pi(\alpha)_{\leq n-1}) \to (\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha), \pi(\alpha)_{\leq n-1})$$

in $\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$*.*

Proof. Note that the morphism $\pi(\alpha)_{\leq n-1}$ is indeed a λ-complete universe in $n\text{-}\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha)$ by Proposition 4.3, since n-categories are closed under limits in $\mathbf{Cat}(\beta)$. Since $\underline{\mathrm{Fam}}_{\pi(\alpha)_{\leq n-1}}$ preserves n-categories, $n\text{-}\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha) \subset \mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha)$ is closed under $\underline{\mathrm{Fam}}_{\pi(\alpha)_{\leq n-1}}$. □

Lemma 8.11. *The coreflection* $\mathcal{X} \mapsto \mathcal{X}_{\leq n-1}$ *defines a morphism*

$$(\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha), \pi(\alpha)) \to (n\text{-}\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha), \pi(\alpha)_{\leq n-1})$$

in $\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$*.*

Proof. Since the coreflection is a right adjoint, it preserves all limits. It remains to show that $\underline{\mathrm{Fam}}_{\pi(\alpha)}(\mathcal{X})_{\leq n-1} \simeq \underline{\mathrm{Fam}}_{\pi(\alpha)_{\leq n-1}}(\mathcal{X}_{\leq n-1})$. Recall that an object of $\underline{\mathrm{Fam}}_{\pi(\alpha)}(\mathcal{X})$ is a pair (I, A) consisting of an object $I \in \mathbf{S}(\alpha)$ and a family of objects $A : I \to \mathcal{X}$. It is $(n-1)$-truncated if and only if I and A_i for any $i \in I$ are $(n-1)$-truncated, that is, it belongs to $\underline{\mathrm{Fam}}_{\pi(\alpha)_{\leq n-1}}(\mathcal{X}_{\leq n-1})$. □

Proof of Theorem 8.2. Let $(\mathcal{C}, u)$ be an object of $\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$ with u $(n-1)$-truncated. By Proposition 8.8, the morphism $!_{\mathcal{C}}$ of Construction 7.7 defines a morphism $(\mathbf{Pr}^{\mathrm{R}}_{\lambda}, \pi(\alpha)_{\leq n-1}) \to (\mathcal{C}, u_{\leq n-1})$. Since u is already $(n-1)$-truncated, $u_{\leq n-1} \simeq u$. We then have a morphism by composition

$$(n\text{-}\mathbf{Pr}^{\mathrm{R}}_{\lambda}, \pi(\alpha)_{\leq n-1}) \to (\mathbf{Pr}^{\mathrm{R}}_{\lambda}, \pi(\alpha)_{\leq n-1}) \xrightarrow{!_{\mathcal{C}}} (\mathcal{C}, u).$$

The uniqueness follows because $\mathcal{X} \mapsto \mathcal{X}_{\leq n-1}$ is a coreflection and is a morphism $(\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha), \pi(\alpha)) \to (n\text{-}\mathbf{Pr}^{\mathrm{R}}_{\lambda}(\alpha), \pi(\alpha)_{\leq n-1})$ in $\mathbf{Cat}_{\mathrm{Lex}(\alpha),\mathrm{Univ}(\lambda)}(\beta)$. □

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