Towards modular proof of normalization for type theories
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Towards modular proof of normalization for type theories

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Normalization (by evaluation)

Challenge

Prove that every type or term in your type theory has a unique normal form.

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Normalization by evaluation is cool (Berger and Schwichtenberg 1991; Altenkirch, Hofmann, and Streicher 1995; Fiore 2002; Coquand 2019; Sterling and Angiuli 2021; Gratzer 2022). But...

- A little bit ad-hoc.
- Proved individually for each type theory.
- Proofs are not modular (at a formal level).

Modular normalization proof

Goal

Prove normalization in a modular/compositional way: a proof of normalization for a complex type theory is composed of proofs of normalization for its smaller fragments.

Modular normalization proof

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For example, consider normalization for the dependent type theory with Π-types and Σ-types.

- The normalization proof is essentially a construction of a special model.
- Π-part and Σ-part of the construction are kind of separate.
- So the whole construction should be the "composite" of its Π-part and Σ-part.

Let's make it formal.

Modular normalization proof

Strategy

Relativize the normalization property.

1. Choose a “correct” category of type theories.
2. Define normalization as a property of a morphism of type theories.
3. Show that the class of morphisms having the normalization property is closed under (transfinite) *composition* and *pushout*.

Then normalization for a “cell complex” follows from normalization for its “cells”.

We’d still need some effort to prove normalization for “cells”, but relativization makes those proofs *reusable* and *composable*, formally.

Outline

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Notion of type theory

Criteria for the notion of type theory.

- A lot of practical examples.
- Essentially algebraic (for colimits of type theories to exist).
- Can talk about normal forms.

Category with representable maps

Category with representable maps (CwR) (Uemura 2019) is a notion of type theory.

- A lot of examples (e.g. Martin-Löf type theory, cubical type theory, two-level type theory, and their fragments and variants).
- Essentially algebraic.

Category with representable maps

Category with representable maps (CwR) (Uemura 2019) is a notion of type theory.

- A lot of examples (e.g. Martin-Löf type theory, cubical type theory, two-level type theory, and their fragments and variants).
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However, it can never speak about normal forms.

- Everything in a CwR is stable under substitution (by design).
- Normal forms are NOT stable under substitution. E.g. $f(0)$ is a normal form for a variable $f : \mathbb{N} \to \mathbb{N}$ but $f(0)[(\lambda x.x)/f] = (\lambda x.x)(0)$ is not.
- Normal forms are stable under *renaming* of variables.

Multiple modes

Idea

A type theory must be able to have multiple "modes".

- Ordinary terms are defined in "substitution mode".
- Normal forms are defined in "renaming mode".

Cf. Multimodal dependent type theory (Gratzer et al. 2021) and its use in normalization proof (Bocquet, Kaposi, and Sattler 2021).

Multimode category with representable maps

Definition

A *multimode category with representable maps* (MCwR) $\mathcal{C}$ consists of:

- a *2-fibration* $\mathbf{E}(\mathcal{C}) \to \mathbf{B}(\mathcal{C})$ (Hermida 1999; Buckley 2014);
- a CwR structure on every fiber (in particular, fibers are 1-categories);
- (one more structure)

and satisfies some axioms (TBD).

- The base 2-category $\mathbf{B}(\mathcal{C})$ is a 2-category of *modes* or a *mode theory*.
- The CwR structure on a fiber $\mathbf{E}(\mathcal{C})_b$ specifies *mode-local rules*.

Multimode type theories

Example

The codomain fibration $\mathbf{DFib} \to \mathbf{Cat}$ carries a canonical structure of MCwR.

Definition

A *multimode type theory* is a small MCwR.

Definition

Let $\mathbb{T}$ be a multimode type theory. A *model* of $\mathbb{T}$ is a morphism $\mathbb{T} \to \begin{pmatrix} \mathbf{DFib} \\ \downarrow \\ \mathbf{Cat} \end{pmatrix}$ of MCwRs. We have a category $\mathbf{Mod}(\mathbb{T})$ of models of $\mathbb{T}$.

Multimode type theories

A model of T is thus a commutative square

$$\begin{array}{ccc} \mathbf{E}(T) & \overset{M}{\dashrightarrow} & \mathbf{DFib} \\ \downarrow & & \downarrow \\ \mathbf{B}(T) & \underset{C}{\dashrightarrow} & \mathbf{Cat}. \end{array}$$

- $C(b)$ for $b \in \mathbf{B}(T)$ is a category of *contexts in mode* b.
- $M(x)$ for $x \in \mathbf{E}(T)_b$ is a discrete fibration ($\simeq$ presheaf) over $C(b)$ of types or terms or something else, depending on what x represents.

Multimode type theories

Example

Every type theory is a multimode type theory with $\mathbf{B}(\mathbb{T}) = \mathbf{1}$.

Example

For any multimode type theory $\mathbb{T}$, there is a *multimode type theory of arrows* $[1] \bullet \mathbb{T}$ such that $\mathbf{Mod}([1] \bullet \mathbb{T}) \simeq \mathbf{Mod}(\mathbb{T})^{\rightarrow}$.

Multimode type theories

Example

There is a type theory $\mathbb{D}$ whose models are categories with families (CwFs). Then $[1] \bullet \mathbb{D}$ has modes $\{0 \to 1\}$, and its models are (pseudo)morphisms of CwFs which look like

$$\begin{array}{ccc} \mathbf{Tm}_0 & \longrightarrow & \mathbf{Tm}_1 \\ \downarrow & & \downarrow \\ \mathbf{Ty}_0 & \longrightarrow & \mathbf{Ty}_1 \\ \downarrow & & \downarrow \\ \mathbf{Ctx}_0 & \longrightarrow & \mathbf{Ctx}_1. \end{array}$$

Type theory with renaming mode

Example (Renaming Type Theory)

There is a multimode type theory $\mathbb{R}$ whose models are models of $[1] \bullet \mathbb{D}$ such that $\mathbf{Ty}_0 \simeq \mathbf{Ty}_1 \times_{\mathbf{Ctx}_1} \mathbf{Ctx}_0$.

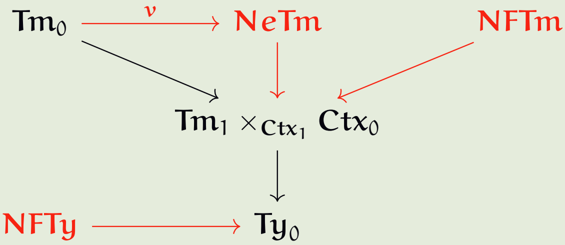
$$
\begin{array}{ccc}
\mathbf{Tm}_0 & \longrightarrow & \mathbf{Tm}_1 \\
\downarrow & & \downarrow \\
\mathbf{Ty}_0 & \longrightarrow & \mathbf{Ty}_1 \\
\downarrow & \lrcorner & \downarrow \\
\mathbf{Ctx}_0 & \longrightarrow & \mathbf{Ctx}_1
\end{array}
$$

The left CwF has the same type as the right CwF, but elements of $\mathbf{Tm}_0$ are thought of as *variables*. Mode 0 is the *renaming mode*. Mode 1 is the *substitution mode*.

Type theory with normal forms

Example (Normal Form Type Theory)

There is a multimode type theory $\mathbb{NF}$ whose models are models of $\mathbb{R}$ equipped with the following additional structures in the renaming mode.



Normalization

Definition

A model of $\mathbb{NF}$ is *normalizing* if

1. $\mathbf{NFTy} \simeq \mathbf{Ty}_0$ (every type has a unique normal form); and
2. $\mathbf{NFTm} \simeq \mathbf{Tm}_1 \times_{\mathbf{Ctx}_1} \mathbf{Ctx}_0$ (every term has a unique normal form).

Definition

Let $\mathbb{T}$ be a multimode type theory extending $\mathbb{NF}$. We say $\mathbb{T}$ is *normalizing* if its initial model is normalizing.

The category of type theories

The correct category of type theories is $\mathbf{MTT}$, the category of multimode type theories.

- A lot of examples (include all CwRs).
- Essentially algebraic.
- Can talk about normal forms.

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Normalizing cover

Definition

Let $\mathbb{T}$ be a multimode type theory extending $\mathbb{NF}$ and let M be a model of $\mathbb{T}$. A *normalizing cover* of M is a morphism $N \to M$ of models of $\mathbb{T}$ with N normalizing.

Proposition

Normalizing models of $\mathbb{NF}$ are closed under retract by definition. □

Corollary

If every model of $\mathbb{T}$ has a normalizing cover, then $\mathbb{T}$ is normalizing. □

Canonical normalizing cover

Let $\mathbb{M}$ be a model of $\mathbb{NF}$. We can construct a *canonical normalizing cover*

$$\mathbf{Nml}(\mathbb{M}) \to \mathbb{M}.$$

The idea is that the *proof-relevant logical relation* (e.g. Coquand 2019) can be turned into a normalizing model.

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Relative normalization property

Definition

Let $f : T_1 \to T_2$ be a morphism between multimode type theories extending $\mathbb{NF}$. We say f *lifts canonical normalizing covers* if the following lift exists.

$$
\begin{array}{ccccc}
\exists N' & \cdots\cdots\to & M & \qquad & \mathbf{Mod}(T_2) \\
\vdots\downarrow & & \downarrow & & \downarrow f^* \\
N & \longrightarrow & M|_{T_1} & & \mathbf{Mod}(T_1) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{Nml}(M|_{\mathbb{NF}}) & \longrightarrow & M|_{\mathbb{NF}} & & \mathbf{Mod}(\mathbb{NF})
\end{array}
$$

Corollary

If $\mathbb{NF} \to T$ lifts canonical normalizing covers, then T is normalizing. □

Stability of normalization property

Proposition

The class of morphisms in $\mathbf{MTT}_{\mathbb{NF}/}$ *that lift canonical normalizing covers is closed under (transfinite) composition and pushout by definition.* □

Conclusion

- The correct category of type theories $\mathbf{MTT}$ is chosen.
- Normalization is a property of a morphism of type theories.
- The normalization property is stable under composition and pushout.

Remarks.

- Lifts of canonical normalizing covers will be constructed by following the usual construction of logical relations and the proof of the uniqueness of normal forms. *Synthetic Tait computability* (Sterling 2021) and an internal language for diagrams of categories (Uemura 2022) are helpful.
- Relativization would also work for other properties such as canonicity.
- There is an "∞-" version of the whole story (cf. Nguyen and Uemura 2022).

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More detailed definition of MCwR

Definition

A *multimode category with representable maps* (MCwR) $\mathcal{C}$ consists of:

- a *2-fibration* $\mathbf{E}(\mathcal{C}) \to \mathbf{B}(\mathcal{C})$ (Hermida 1999; Buckley 2014);
- a CwR structure on every fiber (in particular, fibers are 1-categories);
- a CwR structure on the category $\mathbf{Sect}_u(\mathcal{C})$ of sections $\begin{array}{ccc} & & \mathbf{E}(\mathcal{C}) \\ & \nearrow & \downarrow \\ [1] & \xrightarrow[u]{} & \mathbf{B}(\mathcal{C}) \end{array}$ for every morphism u in $\mathbf{B}(\mathcal{C})$

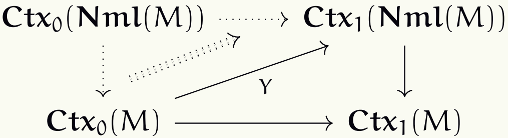
and satisfies some axioms (TBD), e.g.:

- the reindexing $u^* : \mathbf{E}(\mathcal{C})_b \to \mathbf{E}(\mathcal{C})_{b'}$ preserves finite limits for any $u : b' \to b$;
- the restrictions $\mathbf{Sect}_u(\mathcal{C}) \to \mathbf{E}(\mathcal{C})_{b'}$ and $\mathbf{Sect}_u(\mathcal{C}) \to \mathbf{E}(\mathcal{C})_b$ are morphisms of CwRs for any $u : b' \to b$.

Canonical normalizing cover

Let M be a model of $\mathbb{NF}$. We construct a *canonical normalizing cover* $\mathbf{Nml}(M) \to M$. Let M_1 denote the restriction to mode 1.

1. Use the *proof-relevant logical relation* (e.g. Coquand 2019) to get a morphism $\mathbf{Nml}(M)_1 \to M_1$ of models of $\mathbb{D}$ and a functor $Y : \mathbf{Ctx}_0(M) \to \mathbf{Ctx}_1(\mathbf{Nml}(M))$ over $\mathbf{Ctx}_1(M)$.
2. Take the lax limit of Y in $\mathbf{Cat}_{/\mathbf{Ctx}_1(M)}$.



3. This square can be extended to a morphism of models of $\mathbb{NF}$, and $\mathbf{Nml}(M)$ is shown to be normalizing.